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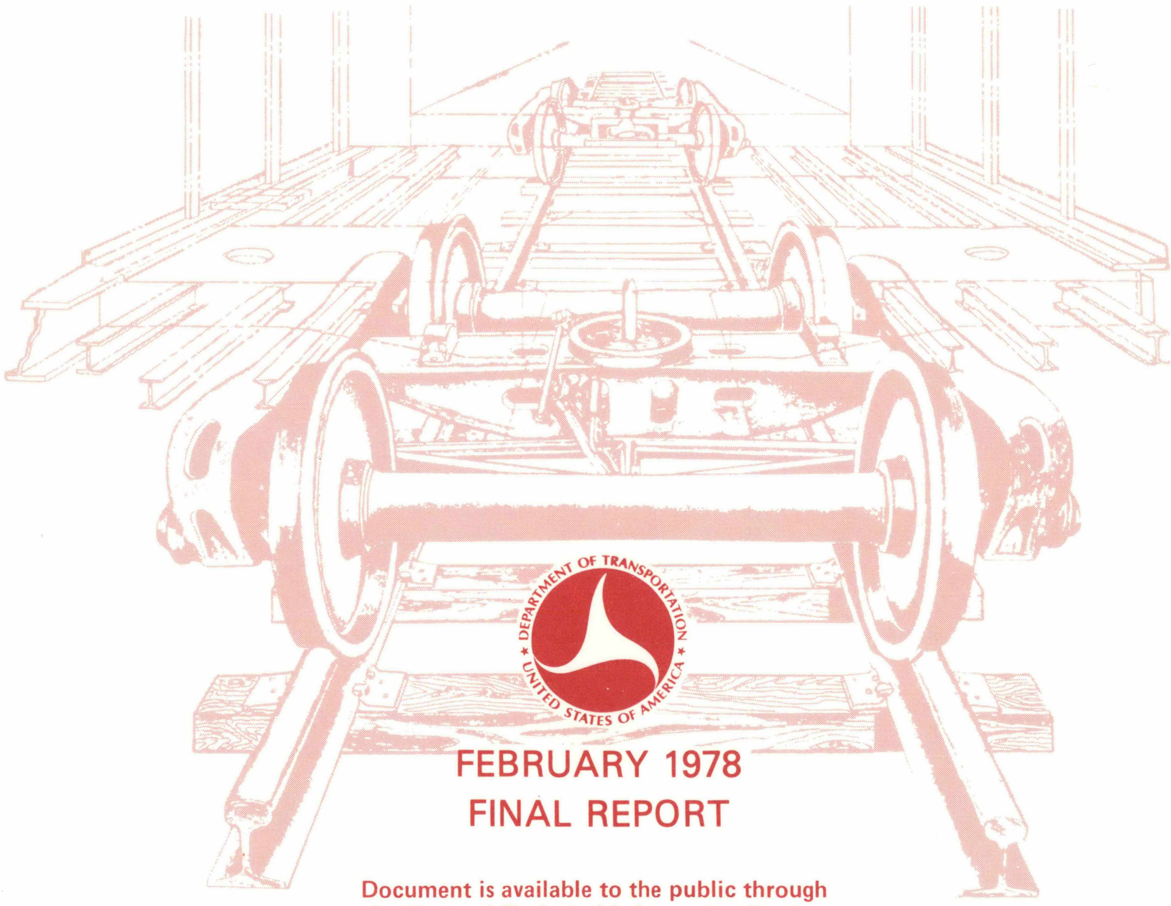
FREIGHT CAR TRUCK DESIGN OPTIMIZATION

VOLUME V — CRITIQUE OF FREQUENCY DOMAIN MODEL — EQUATIONS OF MOTION



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FINAL REPORT

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16. Abstract The mathematical model developed by the Southern Pacific Transportation Co. (SPTCo) during the course of Phase I of the Truck Design Optimization Project (TDOP) is discussed. The equations of motion for a slightly simpler model with simpler nomenclature, which resulted from MITRE/METREK's review of the SPTCo model, are developed and presented. See also:

Vol. I - Executive Summary, FRA/ORD-78/12.I
Vol. II - Phase I Final Report, FRA/ORD-78/12.II
Vol. III - Phase I Frequency Domain Model, FRA/ORD-78/12.III
Vol. IV - Critique of Frequency Domain Model - Solution Techniques, FRA/ORD-78/12.IV
Vol. VI - Critique of Phase I - Test Series Results Reports, FRA/ORD-78/12.VI

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Vol. IX - Results Report for Test Series 4, FRA/ORD-78/12.IX
Vol. X - Results Report for Test Series 3 and 5, FRA/ORD-78/12.X
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1.0 INTRODUCTION

As part of the Truck Design Optimization Project (TDOP) the contractor, Southern Pacific Transportation Company (SP), was required to develop a mathematical model to be used for predicting truck behavior. Such a model was developed during the later portions of the Phase I effort of the program. The model, its description, the nomenclature used, and the development of the equations of motion were all oriented toward the computer program which would ultimately solve the equations.

As a part of MITRE/METREK's work for FRA on the TDOP program, METREK performed a detailed review and evaluation of the SP mathematical model. This review was judged necessary to provide an independent evaluation of the model which contained many assumptions and some errors whose effect was not clear.

This paper documents our review of the contractor's model and develops the equivalent equations. The Laplace transform variable "s" is used throughout instead of the frequency variable "ω", and certain terms have been omitted from the equations in the interest of simplicity, following the example of several authors [1], [2], [4], [6].

Otherwise, no attempt has been made to distinguish the model from the contractor's effort. Of necessity, to avoid the somewhat cumbersome computer-oriented nomenclature of the contractor's model, the nomenclature is different, but the number of degrees of freedom is the same, the coordinate systems are the same, the degrees of freedom are identical with the ones chosen by the contractor, and the generalized coordinates have been chosen to be identical with those chosen by the contractor, even in instances where another choice might have seemed preferable.

One distinction between the model described herein and the contractor's model is that this model is completely linear, with velocitydependent damping, whereas the contractor's model is quasi-linear in that non-linear frictional damping has been approximated by frequency and amplitude dependent coefficients. It was felt in developing the equations of motion for the model herein that the relative simplicity of the completely linear model would outweigh the possible greater accuracy obtained by the use of a better approximation of damping. Another distinction is that damping of at least some magnitude has been assumed in this model for all springs, including structural stiffnesses, whereas the contractor's model does not assume damping for all such springs.

Since the model is completely linear, the equations of motion are developed by means of the Newtonian method, although in several cases the equations were checked by means of the Lagrangian method.

2.0 MODEL DESCRIPTION

The coordinates are defined as follows:

Translational:

x, parallel to track centerline, positive in direction of motion of vehicle;

y, perpendicular to track centerline in the horizontal plane, positive to the right when facing the direction of motion of the vehicle;

z, positive upwards.

Rotational:

α, pitch angle, positive with front end of vehicle up, rear end down.

θ, roll angle, positive clockwise when facing direction of motion of vehicle;

ψ, yaw angle, positive clockwise looking down on vehicle.

These coordinates are indirectly illustrated in Figures 1, 2, and 3, which show the lumped masses and the springing and damping between them which constitute the linear model.

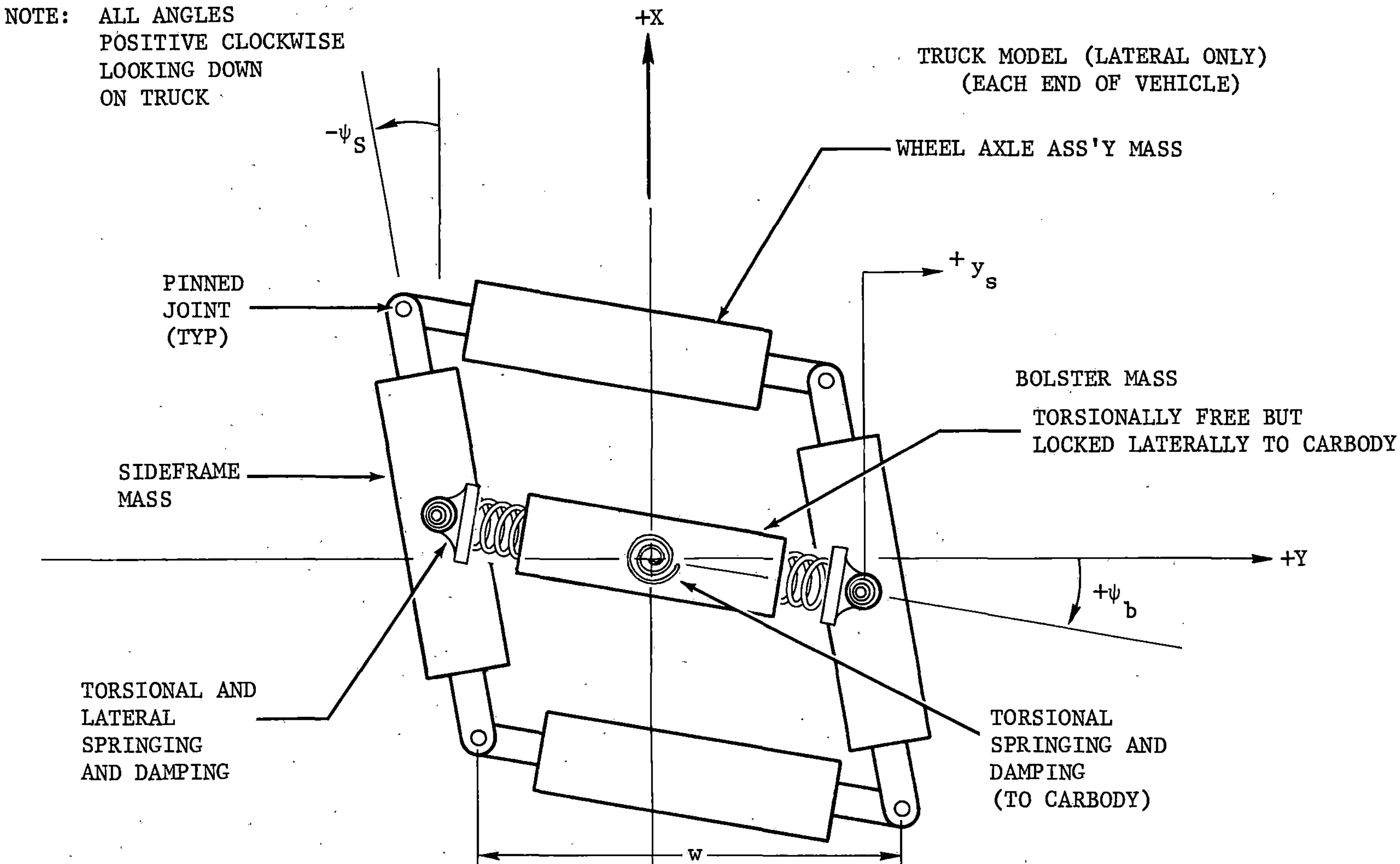
The model has a total of thirteen degrees of freedom: three lateral degrees of freedom for each of two trucks, two lateral degrees of freedom for the entire carbody as a single mass, and five degrees of freedom associated with the division of the carbody into two separate masses in considering roll motions and motions in the vertical plane. The terminology used for the generalized coordinates associated with these degrees of freedom is as follows:

ψ_b = absolute yaw angle of bolster

ψ_s = absolute yaw angle of side frame

y_s = absolute lateral displacement of side frame

ψ_c = absolute yaw angle of entire carbody

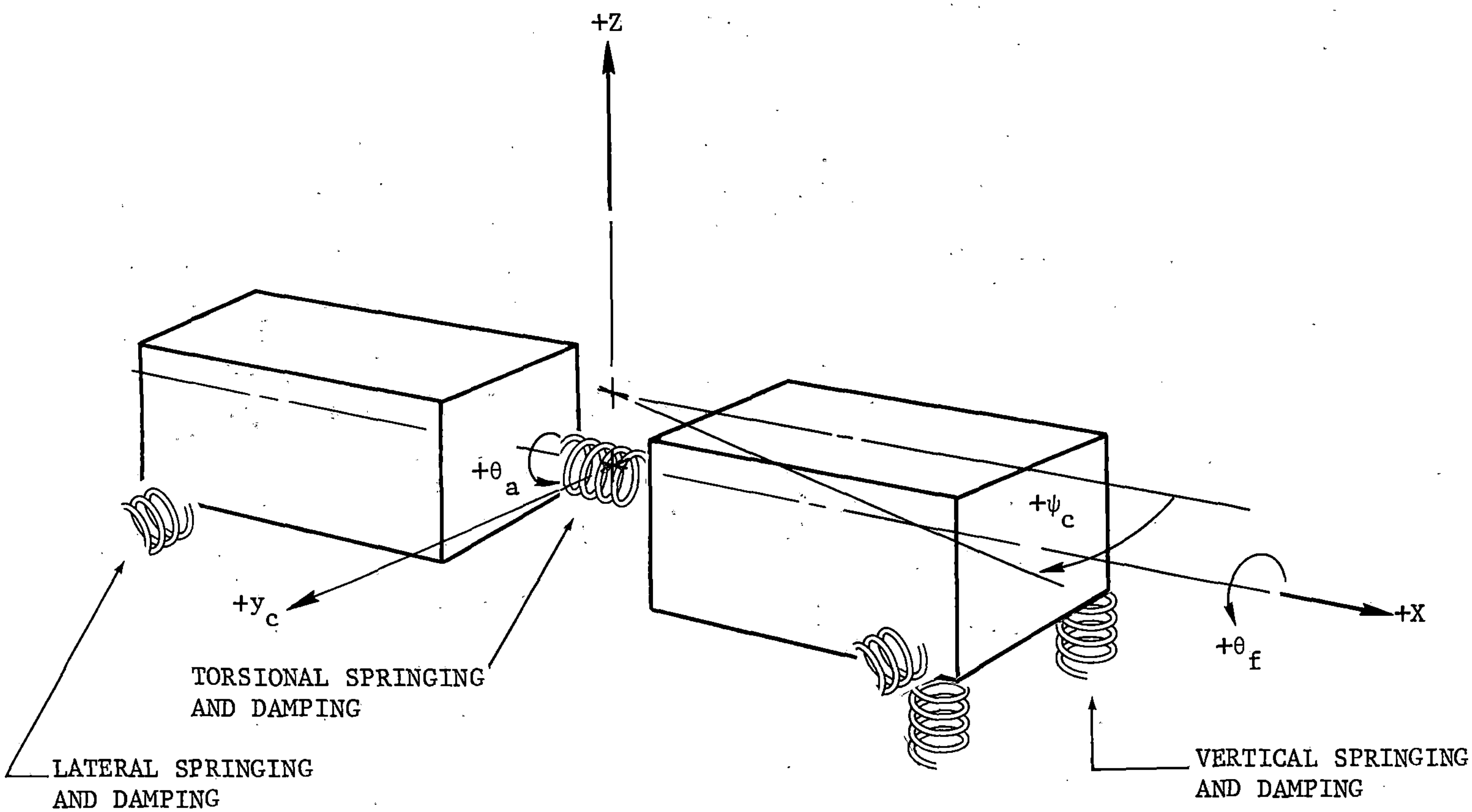


DEGREES OF FREEDOM: (1) ψ_b, YAW ANGLE OF BOLSTER

(2) ψ_S, YAW ANGLE OF SIDEFRAME

(3) y_s, LATERAL DISPLACEMENT OF SIDEFRAME

FIGURE 1. TRUCK MODEL



DEGREES OF FREEDOM:

(1) ψ_c, YAW ANGLE OF ENTIRE CARBODY

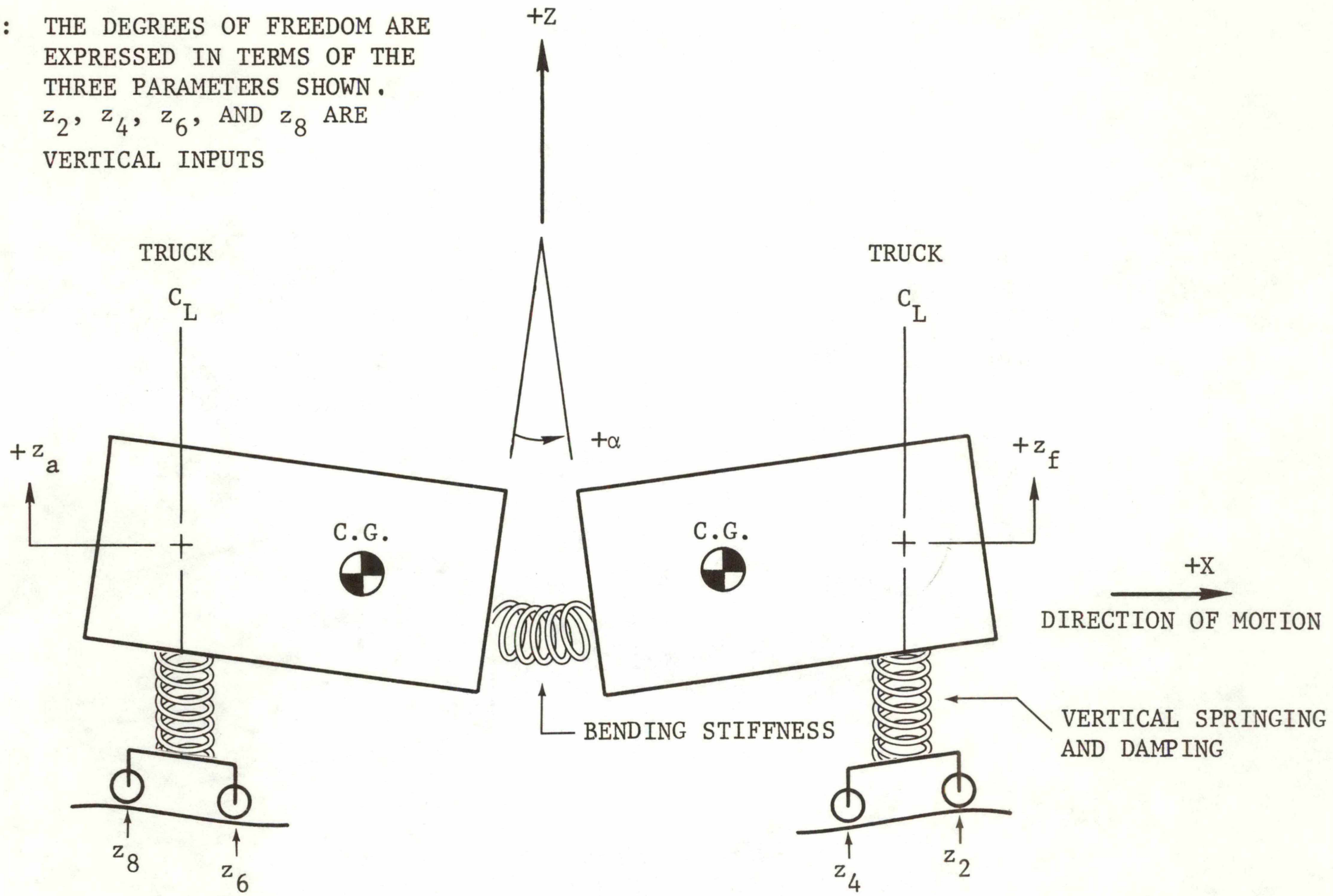
(2) y_c, LATERAL DISPLACEMENT OF ENTIRE CARBODY

(3) θ_f, roll angle of fore carbody section

(4) θ_a, roll angle of aft carbody section

FIGURE 2. CARBODY LATERAL/VERTICAL MODEL

NOTE: THE DEGREES OF FREEDOM ARE EXPRESSED IN TERMS OF THE THREE PARAMETERS SHOWN. z_2, z_4, z_6, AND z_8 ARE VERTICAL INPUTS



DEGREES OF FREEDOM: (1) PITCH
(2) HEAVE
(3) FIRST BENDING MODE

FIGURE 3. CARBODY VERTICAL MODEL

y_c = absolute lateral displacement of entire carbody

θ_f = roll angle of fore carbody section

θ_a = roll angle of aft carbody section

z_f = absolute vertical displacement of fore carbody section at centerplate

z_a = absolute vertical displacement of aft carbody section at centerplate

α = relative pitch angle between fore and aft carbody sections

Note that:

(1) z_f and z_a refer to displacements not of the center of gravity (C.G.) of the section but of the end of the carbody section directly over the centerplate;

(2) the last three generalized coordinates specify what are essentially the three following vertical degrees of freedom for the carbody as a whole: pitch, heave, and first bending mode.

The appropriate additional subscripts "f" and "a" will be used for the first three coordinates above where necessary to distinguish between the fore and aft trucks. The use of a relative pitch angle for the final coordinate requires a further definition, as below:

$$\alpha = \alpha_f - \alpha_a \tag{1}$$

where

α_f = absolute pitch angle of fore carbody section;

α_a = absolute pitch angle of aft carbody section.

It is also assumed implicitly that the side frame position, both vertically and laterally, and the position of the corresponding wheels are functionally dependent. In considering lateral motion of the truck, the wheels and side frames are assumed to be longitudinally in line, and the dimension "w" is used for the separation between the two sides. Thus, the yaw angle of the side frame is the difference between the lateral displacements of the fore and aft wheels divided by the distance between them. In considering vertical motion of the truck, the wheels are considered to be longitudinally in line with the vertical spring groups acting in the bolster, and the dimension "b" is used for the lateral separation between these spring groups. The vertical position of the bottom of these spring groups acting on the bolster is assumed to be the average of the vertical displacements of the wheels on that side.

In this linear model it is also assumed that the wheel does not leave the rail; i.e., there is no wheel hop. Thus, the forces at the wheel-rail interface and the geometry of the interface are continuous functions of time, and both the forces at the interface and the lateral and vertical positions of the wheels can be formulated to supply continuous forcing functions for the model.

A word of discussion about each illustration will clarify aspects of the model which might otherwise not be evident. The truck model is constrained by what can be considered pinned joints at the six junctures so that the bolster and the two wheel-axle assemblies always move in parallel; likewise the two side frames. The four joints at the corners are assumed to be ideal frictionless joints, which transmit no moments, only forces along the longitudinal axes of symmetry of the particular masses. In contrast, the joints at the junctions of the bolster and side frames are constrained by torsional springing and damping. In addition, the bolster may move laterally between these support points, in which motion it is locked to the carbody section above it, so that lateral motion of the end of the carbody is identical with lateral motion of the bolster. However, the bolster may move in yaw with respect to this same carbody, although it is constrained by torsional springing and damping acting between these bodies.

The effect of the assumed geometry and mechanical constraints is that in considering the yaw mode of the side frame a term must be introduced to account for the inertial moment of the wheel-axle assembly; correspondingly, in considering the yaw mode of the bolster, a term must be introduced to account for the inertial moment of the side frames. Also, in the lateral mode of side frame motion the masses of the wheel-axle assemblies must be added to the mass of the side frames.

It should also be noted that in this truck model complete mechanical and geometrical symmetry is assumed; the side frames are identical, and the wheel-axle assemblies are identical, in dimensions, mass, and moments of inertia. If such symmetry is not assumed, additional terms, which effectively couple the yaw and lateral modes of side frame motion, must be introduced. Furthermore, it is also assumed in the overall model that identical trucks are used on each end of the carbody. However, in the development of the equations, provision is made for different spring groups and damping coefficients in the fore and aft carbody sections and on the left and right sides of the bolsters. Such differences will effectively couple the vertical and roll modes and the lateral and yaw modes.

The second figure illustrates that the model permits each section of the carbody to roll independently, with the constraint of torsional springing and damping between them. The roll constraint is supplied by the spring groups and effective dashpots at each end of the bolster. The lateral springing and damping shown is the same as that shown between the bolster and side frames on the previous illustration. Note that although the carbody is split in two as far as roll is concerned, in the yaw mode and lateral mode the carbody is considered as a single unit.

The third figure illustrates what are effectively the pitch, heave, and first bending modes for the carbody in the vertical plane. In view of the decision to model the first bending mode in such a fashion, the choice by the contractor of the three generalized coordinates seemed appropriate, and MITRE followed his choice. The z_2 through z_8 are the vertical positions of the wheels on the right side, and vertical forces are introduced into the carbody through the vertical springing and damping between the bolster and side frame, the position of the center of the side frame being the average of the corresponding wheel positions.

Note that the three generalized coordinates completely define the geometry of the bodies with respect to the three degrees of freedom permitted, although other choices could have been made for the coordinates. Effectively the two carbody sections are constrained at the center so that relative vertical motion between them at that point is not permitted, although relative angular motion, as defined by the angle α, is permitted. Also note that while the generalized coordinate corresponding to vertical displacement is measured over the centerplate, which lies over the springs shown in the figure, the C.G. of the section lies somewhere between the centerplate and the center of the entire carbody. Its distance from the centerplate is designated by x_{ca} and x_{cf}, and these two parameters may be different.

3.0 DERIVATION OF THE EQUATIONS OF MOTION

3.1 Truck Equations

The equations of motion for the truck will be derived first. These will be developed for a single truck, and subsequently appropriate subscripts for "fore" and "aft" trucks will be incorporated into the general equations.

In this truck model as shown, there are inherently four degrees of freedom: lateral motion of the side frames and bolster, and yaw motion of the side frames and bolster; and the equations of motion will be developed in a corresponding fashion. However, it is recognized that the constraint that the bolster is locked to the carbody in the lateral mode has been assumed. This will require that what is referred to in the development of the truck equations as the lateral displacement of the bolster is actually the lateral displacement of the carbody as well, and such an accommodation will be made at a later stage in the development of the equations. Figure 4 illustrates the two possible yaw modes.

Consider first the simplest mass, the bolster, and let the following forces be defined:

F_c = lateral force on bolster from carbody;

F_1 = lateral force on bolster due to springing and damping at left side;

F_2 = lateral force on bolster due to springing and damping at right side.

All forces are positive in the direction shown on the free body diagram, Figure 5.

Then, assuming zero initial conditions to permit immediate use of Laplace transformed equations, summation of the forces acting on the bolster results in

$$F_1 + F_c - F_2 = m_5 s^2 y_5 \qquad (2)$$

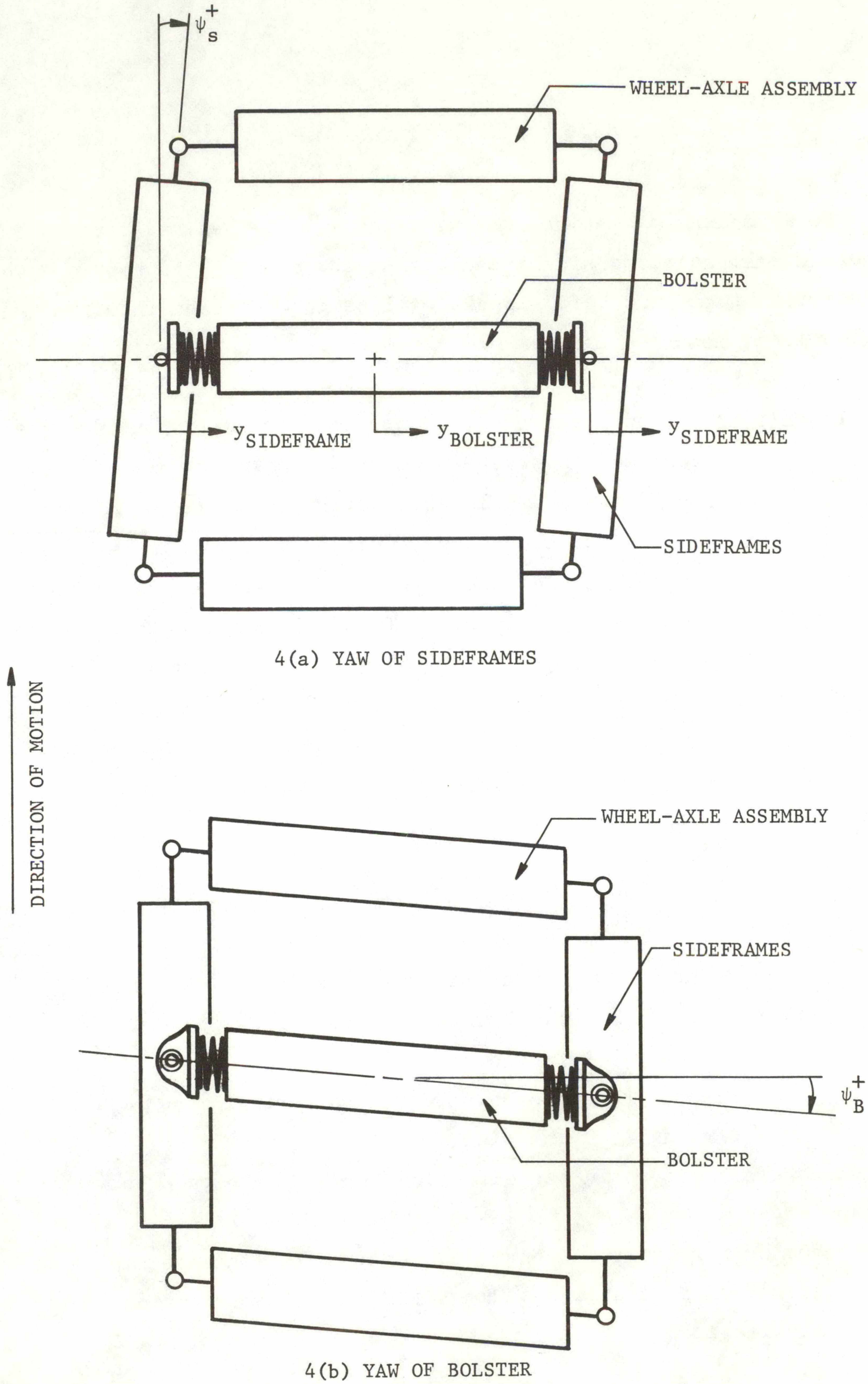


FIGURE 4. TRUCK YAW MODES

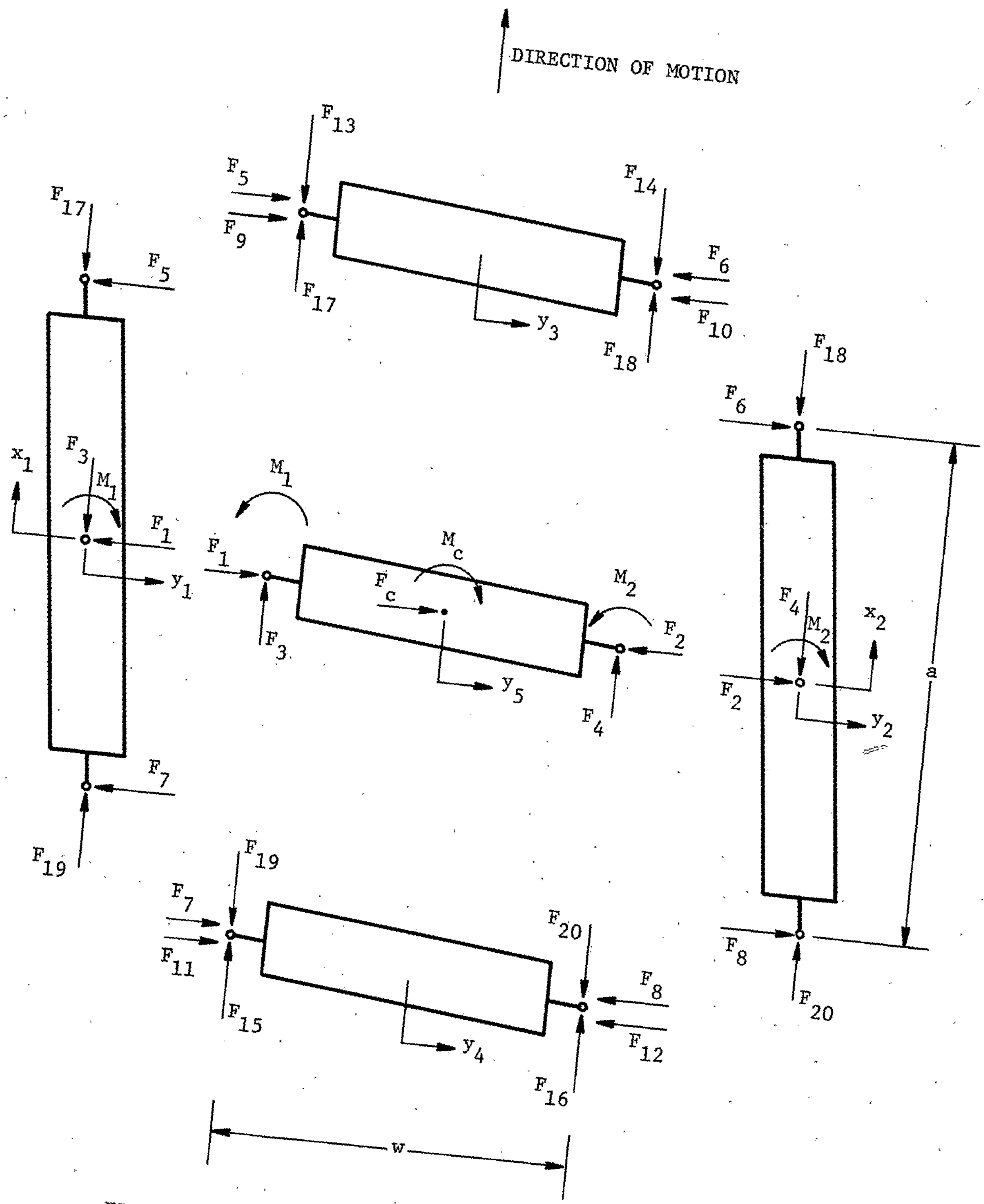


FIGURE 5. TRUCK COMPONENT FREE BODY DIAGRAM

But by definition,

$$F_1 = (C_1 s + K_1)\ (y_1 - y_5) \tag{3a}$$

$$F_2 = (C_2 s + K_2)\ (y_5 - y_2) \tag{3b}$$

Substitution of these expressions into (2) above results in:

$$F_c + (C_1 s + K_1)y_1 + (C_2 s + K_2)y_2 = \tag{4}$$

$$[m_5 s^2 + (C_1 + C_2)s + (K_1 + K_2)]y_5$$

Now it can be seen from the geometry of the configuration that for small deflections y_1 must equal y_2. Furthermore, let it be assumed that the lateral springing and damping on each side are identical, so that

$$C_1 = C_2 \text{ and } C_\ell \triangleq C_1 + C_2 \tag{4a}$$

and

$$K_1 = K_2 \text{ and } K_\ell \triangleq K_1 + K_2 \tag{4b}$$

Then calling the lateral motion of the side frames y_s and using the combined springing and damping coefficients K_ℓ and C_ℓ, (4) becomes

$$F_c + (C_\ell s + K_\ell)y_s = (m_5 s^2 + C_\ell s + K_\ell)y_5 \tag{5}$$

Consider now the summation of lateral forces acting on the side frames. It must be noted that what we choose to call external forces act only on the wheel-axle assemblies at the wheel-rail interface and do not act directly on the side frame. However, at each wheel-rail interface, a lateral and a longitudinal external force and an external moment are introduced; the moment, however, has been shown to have a negligible effect on truck dynamics [6] and will be neglected in this model.

Thus, referring to the free body diagram for the left side frame with F_1 in an equal and opposite sense to its previous application, we may write:

$$- F_1 - F_5 - F_7 = m_1 s^2 y_1 \qquad (6)$$

Similarly for the right side frame

$$F_2 + F_6 + F_8 = m_2 s^2 y_2 \qquad (7)$$

Note that although moments and longitudinal forces act on the wheel-axle assemblies in addition to lateral forces, only the lateral forces are transmitted to the side frames in a fashion to give lateral accelerations to the side frame masses. Moments and longitudinal forces applied to the wheel-axle assemblies both react in the side frame as longitudinal forces. Then for the fore wheel-axle assembly we may write:

$$F_5 + F_9 - F_6 - F_{10} = m_3 s^2 y_3 \qquad (8)$$

and similarly for the aft assembly:

$$F_7 + F_{11} - F_8 - F_{12} = m_4 s^2 y_4 \qquad (9)$$

where F_9, F_{10}, F_{11}, and F_{12} are the lateral external forces applied to the wheel-axle assemblies at the wheel-rail interface.

Now it has been noted that

$$y_1 = y_2 \triangleq y_s \qquad (10)$$

and for small angles of yaw of the side frames it is seen that

$$y_3 = y_1 + \frac{a}{2}\psi_s \qquad (11)$$

$$y_4 = y_1 - \frac{a}{2}\psi_s \qquad (12)$$

Therefore we have, adding (6) and (7), rearranging, and substituting (3a), (3b), (10), (11), and (12) into the sum of (8) and (9), after some algebraic manipulation:

$$\begin{aligned} F_9 - F_{10} + F_{11} - F_{12} = \Big\{ & [(m_1 + m_2) + (m_3 + m_4)]s^2 + \\ & (C_1 + C_2)s + (K_1 + K_2)\Big\} y_s \qquad (13) \\ & -[(C_1 + C_2)s + (K_1 + K_2)]y_5 \\ & + (m_3 - m_4)\frac{a}{2} s^2 \psi_s \end{aligned}$$

If as before we assume the lateral spring and dashpot constants are equal, we may write:

$$F_9 - F_{10} + F_{11} - F_{12} = [(m_1 + m_2 + m_3 + m_4)s^2 + C_\ell s + K_\ell]y_s \qquad (14)$$

$$- (C_\ell s + K_\ell)y_5 + (m_3 - m_4)\ \frac{a}{2}\ s^2\psi_s$$

Note that the masses of the side frames and wheel-axle assemblies add directly in the first term on the right, but that in the last term it is the difference of the masses of the wheel-axle assemblies which is significant.

If the masses are symmetrically identical, and we define

$$m_s \stackrel{\Delta}{=} m_1 + m_2 + m_3 + m_4 \qquad (15)$$

we obtain

$$F_9 - F_{10} + F_{11} - F_{12} = (m_s s^2 + C_\ell s + K_\ell)y_s - (C_\ell s + K_\ell)y_5 \qquad (16)$$

as the equation for lateral motion of the side frames.

Consider now the summation of moments acting on the side frame. The same free body diagrams are used, and recall from page 8 that no moments are transferred to the side frames through the frictionless pinned joints between the side frames and the wheel-axle assemblies (although at the juncture of the side frames and bolster restraining springing and damping have been assumed, resulting in a moment there which must be considered). Longitudinal forces transmitted through these same pinned joints do not contribute to yaw of the side frames, since small angular deflections have been assumed, and the moment arm is a differential quantity.

Thus, returning to the free body diagram of the left side frame we may write:

$$M_1 + (F_7 - F_5)\ \frac{a}{2} = J_1 s^2\psi_s \qquad (17)$$

where M_1 is the restraining moment between bolster and side frame. (It is seen from the assumed geometry that the yaw angle of both side

frames is constrained to be the same; however, at this point, the moments of inertia have not been assumed to be the same).
Similarly for the right side frame:

$$M_2 + (F_6 - F_8)\frac{a}{2} = J_2 s^2 \psi_s \tag{18}$$

Note that no moment is contributed in either case by F_1 or F_2 which act at the presumed center of gravity of the masses. Also note that while there are moments applied to the wheel-axle assemblies at the wheel-rail interface they do not contribute to the lateral motion of, or forces on, the wheel-axle assemblies.

After some algebraic manipulation, and using equations (8) and (9), (11) and (12) and (17) and (18) become, upon addition:

$$M_1 + M_2 + (F_9 - F_{10} - F_{11} + F_{12})\,\frac{a}{2} = [(J_1 + J_2) + (m_3 + m_4)\,\frac{a^2}{4}]\; s^2\psi_s$$
$$+ (m_3 - m_4)\,\frac{a}{2}\,s^2 y_s \tag{19}$$

Define

C_3 = coefficient of linear torsional damping between bolster and side frame (left side)

K_3 = spring constant of linear torsional spring between bolster and side frame (left side)

C_4 = same as C_3, only for right side

K_4 = same as K_3, only for left side.

Then $M_1 = (C_3 s + K_3)\;(\psi_b - \psi_s)$ (20)

$$M_2 = (C_4 s + K_4)\;(\psi_b - \psi_s) \tag{21}$$

where ψ_b = yaw angle of bolster.

With again the assumption of mass symmetry between the wheel-axle assemblies, so that the y_s coupling term drops out, and defining

$$J_s \triangleq J_1 + J_2 + (m_3 + m_4) \frac{a^2}{4} \tag{22}$$

$$C_b \triangleq C_3 + C_4 \tag{23}$$

$$K_b \triangleq K_3 + K_4 \tag{24}$$

the equation for yaw motion of the side frames becomes:

$$(F_9 - F_{10} - F_{11} + F_{12}) \frac{a}{2} = (J_s s^2 + C_b s + K_b) \psi_s - (C_b s + K_b) \psi_b \tag{25}$$

The last truck equation relates to the yaw of the bolster. Moments are contributed at the carbody-bolster interface and at the side frame interfaces. Longitudinal forces also act on the bolster at the ends. Lateral forces acting at those points, because the yaw angle of the bolster is a small quantity, are considered to contribute only a negligible moment and are not considered further.

Defining moments and forces positive as shown, and summing moments, we obtain

$$M_c - M_1 - M_2 + (F_3 - F_4) \frac{w}{2} = J_5 s^2 \psi_b \tag{26}$$

Proceeding to the free body diagram of the left side frame, we obtain, after summation of longitudinal forces:

$$F_{19} - F_3 - F_{17} = m_1 s^2 x_1 \tag{27}$$

The moments on the side frames have already been considered. Similarly for the right side frame:

$$F_{20} - F_4 - F_{18} = m_2 s^2 x_2 \tag{28}$$

Consideration of yaw of the fore wheel-axle assembly leads to:

$$(F_{17} - F_{13} - F_{18} + F_{14}) \frac{w}{2} = J_3 s^2 \psi_b \tag{29}$$

since by the geometric constraint assumed the yaw angle of the wheel-axle assemblies is the same as that of the bolster, and for the aft assembly

$$(F_{15} - F_{19} - F_{16} + F_{20}) \frac{w}{2} = J_4 s^2 \psi_b \tag{30}$$

Define $M_c \triangleq (C_c s + K_c)\ (\psi_c - \psi_b)$ (31)

and recall that

$$M_1 = (C_3 s + K_3)\ (\psi_b - \psi_s) \tag{20}$$

$$M_2 = (C_4 s + K_4)\ (\psi_b - \psi_s) \tag{21}$$

Now from (27) and (28)

$$F_3 = F_{19} - F_{17} - m_1 s^2 x_1 \tag{32}$$

$$F_4 = F_{20} - F_{18} - m_2 s^2 x_2 \tag{33}$$

Substituting into (26), we obtain:

$$\begin{aligned} &(C_c s + K_c)\ (\psi_c - \psi_b) - (C_3 s + K_3)\ (\psi_b - \psi_s) \\ &- (C_4 s + K_4)\ (\psi_b - \psi_s) + F_{19} \frac{w}{2} - F_{17} \frac{w}{2} - m_1 \frac{w}{2} s^2 x_1 \\ &- F_{20} \frac{w}{2} + F_{18} \frac{w}{2} + m_2 \frac{w}{2} s^2 x_2 = J_5 s^2 \psi_b \end{aligned} \tag{34}$$

From geometry,

$$x_1 = \frac{w}{2} \psi_b \tag{35a}$$

$$x_2 = - \frac{w}{2} \psi_b \tag{35b}$$

and recalling (29) and (30) and making the appropriate substitutions, after some algebraic manipulations equation (34) becomes

$$(F_{15} - F_{16} + F_{14} - F_{13}) \frac{w}{2} = \left\{ \left[J_3 + J_4 + J_5 + (m_1 + m_2) \frac{w^2}{4} \right] s^2 + (C_b + C_c)s + (K_b + K_c) \right\} \psi_b - (C_b s + K_b)\, \psi_s - (C_c s + K_c)\, \psi_c \tag{36}$$

If we define

$$J_b \triangleq J_3 + J_4 + J_5 + (m_1 + m_2) \frac{w^2}{4} \tag{37}$$

the final equation for the yaw of the bolster becomes:

$$(F_{15} - F_{16} + F_{14} - F_{13}) \frac{w}{2} = \left[J_b s^2 + (C_b + C_c)s + (K_b + K_c) \right] \psi_b - (C_b s + K_b)\, \psi_s - (C_c s + K_c)\, \psi_c \tag{38}$$

3.2 Body Equations

It should be recalled that the model presumes that the bolster is locked to the carbody laterally, so that the lateral displacement of the bolster is identical with the lateral displacement of the corresponding end of the carbody. While y_5 was used in the derivation of the truck equations to designate the bolster lateral displacement, in this portion of the derivation the corresponding lateral displacements of the ends of the carbody at the bolster level where the lateral spring forces are presumed to act will be determined from the lateral displacement and yaw angle of the carbody as a whole and the roll angle of the corresponding fore or aft body sections. Later, the bolster displacement y_5 will be dropped from the truck equations and the corresponding lateral displacements of the ends of the carbody as discussed above will be substituted in the formulation of the final equations.

The model also provides for spring groups with different vertical spring constants and different equivalent vertical damping coefficients in each side of the bolster. These are introduced into the equations as the sum and difference of the values on each side.

3.2.1 Roll Moments Equations

Figures 6 and 7 illustrate the geometry of the carbody sections and some of the coordinates and variables. In this connection, note that z_f represents the vertical displacement of the carbody end directly above the centerplate and not at the C.G. of the section; in this respect, the end of the carbody beyond the centerplate is effectively ignored. Another variable, z_{cf}, used in the derivations, refers to the vertical displacement at the C.G. of the section.

The distinction between x_f and x_{cf} and between x_a and x_{ca} should be noted, particularly as there is a constraint among these terms and their values cannot be independently designated. The terms with the single subscript, x_f and x_a, refer to the distances from the centerplates to the C.G. of the entire carbody, and their total is equal to the separation between centerplates. The terms with the double subscript refer to the distance from the centerplate to the C.G. of the fore and aft sections. The distances are related to each other and to the masses of the individual carbody sections by the following constraint:

$$m_f \, (x_f - x_{cf}) = m_a \, (x_a - x_{ca})$$

Note also that while the wheel vertical positions form the vertical inputs to this model, the lateral position of the center of the side frame forms the lateral input; this position is derived from the lateral forces acting on the wheel and the lateral wheel positions.

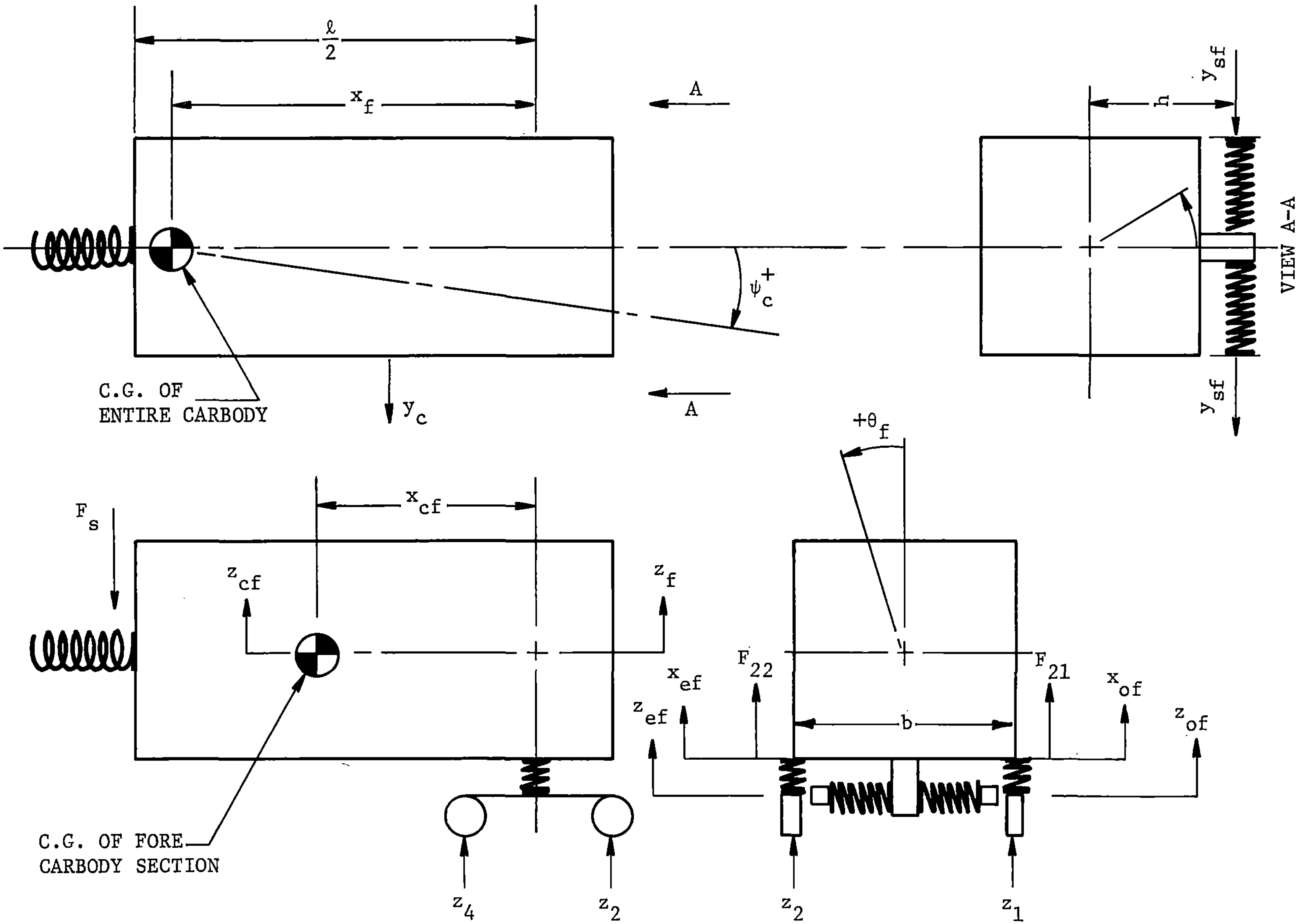
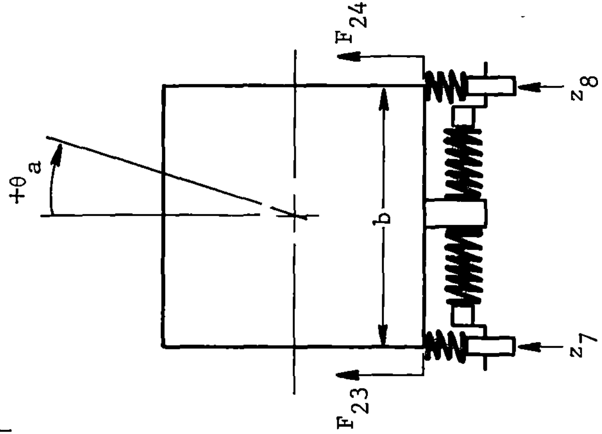
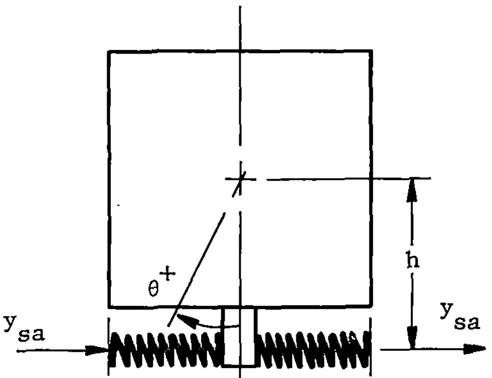


FIGURE 6. FORE CARBODY SECTION





VIEW A-A

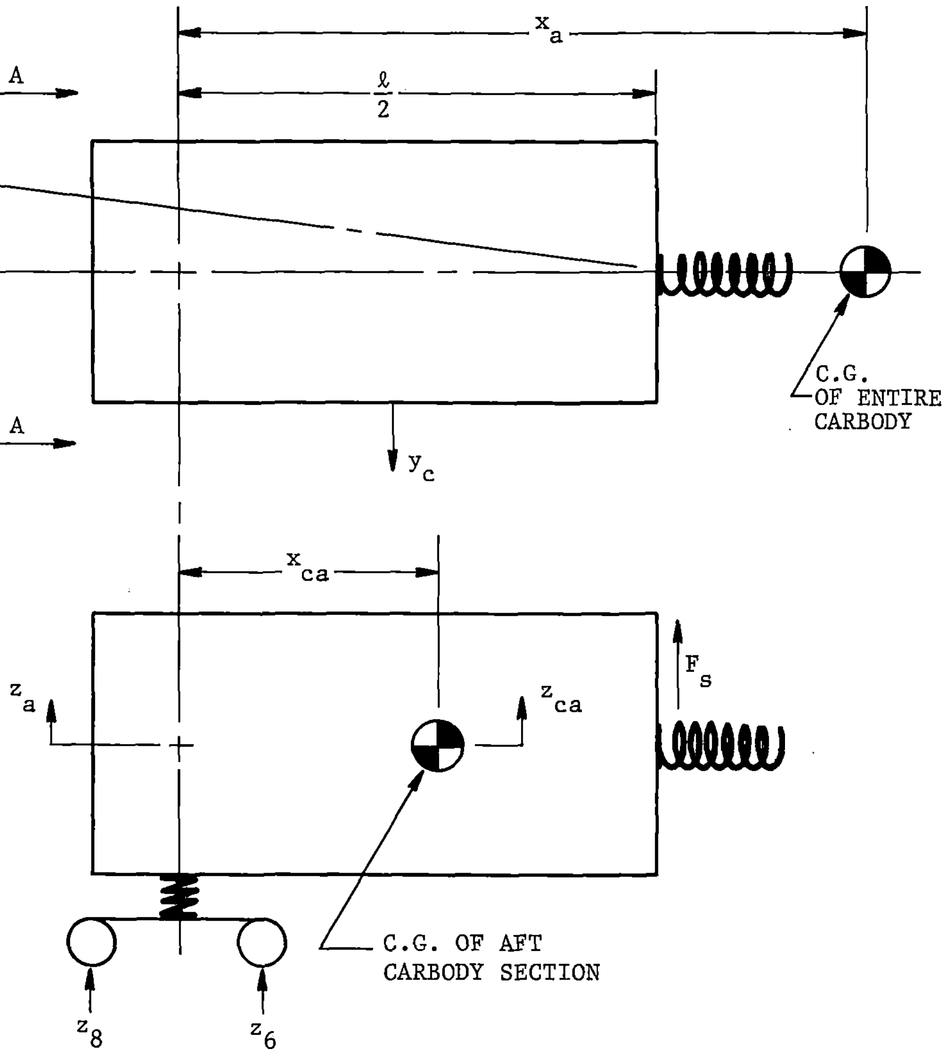


FIGURE 7. AFT CARBODY SECTION

The convention for numerical subscripts for the vertical inputs at the wheels will be that adopted by the contractor, namely, looking down from the top, with the direction of motion toward the top of the page:

1	2
3	4
5	6
7	8

Consider first the roll moments on the fore carbody section. The model provides for different vertical spring groups on each side. Let

K_{of} = spring constant of springs on odd-numbered (left) side

K_{ef} = spring constant of springs on even-numbered (right) side

C_{of}, C_{ef} = corresponding damping terms

for the fore carbody section. Note that throughout the remaining development, appropriate "fore" and "aft" subscripts must be used.

Then define the sum and difference of these spring constants for convenience.

$$K_{sf} = K_{ef} + K_{of} \tag{39a}$$

$$C_{sf} = C_{ef} + C_{of} \tag{39b}$$

$$K_{df} = K_{ef} - K_{of} \tag{40a}$$

$$C_{df} = C_{ef} - C_{of} \tag{40b}$$

Also assume for the moment two fictitious forces acting on the carbody section at the top of the bolster springs, F_{21} and F_{22}, as shown.

The positive roll moment will be

$$\left(F_{21} - F_{22}\right)\frac{b}{2}$$

Defining some intermediate displacement variables x_{ef} and x_{of} as the vertical displacements of the corners of the carbody and z_{ef} and z_{of} as the vertical displacements of the bottom of the spring set on the corresponding corners, we may say

$$F_{22} = \left(C_{ef}s + K_{ef}\right)\left(z_{ef} - x_{ef}\right) \tag{41}$$

and

$$F_{21} = \left(C_{of}s + K_{of}\right)\left(z_{of} - x_{of}\right) \tag{42}$$

But*

$$z_{of} = \frac{z_1 + z_3}{2} \tag{43}$$

$$z_{ef} = \frac{z_2 + z_4}{2} \tag{44}$$

Consequently, the positive roll moment becomes, after substitution:

$$\left(F_{21} - F_{22}\right)\frac{b}{2} = \frac{b}{2}\left[(C_{of}s + K_{of})\left(\frac{z_1 + z_3}{2} - x_{of}\right) - (C_{ef}s + K_{ef})\left(\frac{z_2 + z_4}{2} - x_{ef}\right)\right] \tag{45}$$

*Note that these relationships are only approximately true because of the difference between b and w. However, since the ratio b/w is very nearly unity, the advantages in simplifying the expressions gained by ignoring the distinction seemed to outweigh the slightly greater accuracy obtainable by considering it.

For convenience, the following terms are defined:

$$\theta_f = \frac{x_{of} - x_{ef}}{b} \qquad \text{roll of fore carbody section} \tag{46}$$

$$z_f = \frac{x_{of} + x_{ef}}{2} \qquad \text{vertical displacement of end of fore carbody section over centerplate} \tag{47}$$

$$\theta_F = \frac{z_1 + z_3}{2b} - \frac{z_2 + z_4}{2b} \qquad \text{roll angle of an equivalent single axle, fore carbody section} \tag{48}$$

$$z_{wf} = \frac{z_1 + z_2 + z_3 + z_4}{4} \qquad \text{average vertical input*} \tag{49}$$

The variables in the initial equation (45) can be put into terms of these by means of some intermediate expressions, namely

$$K_{of} = \frac{K_{sf} - K_{df}}{2} \quad \text{and } C_{of} = \frac{C_{sf} - C_{df}}{2} \tag{50}$$

$$K_{ef} = \frac{K_{sf} + K_{df}}{2} \quad \text{and } C_{ef} = \frac{C_{sf} + C_{df}}{2} \tag{51}$$

$$x_{ef} = z_f - \frac{b}{2}\theta_f \tag{52}$$

$$x_{of} = z_f + \frac{b}{2}\theta_f \tag{53}$$

After some algebraic manipulation, the roll moment attributable to vertical inputs can be shown to be (after adding the damping term):

$$\frac{b^2}{4}\left[(C_{sf}\, s + K_{sf})\,(\theta_F - \theta_f) + (C_{df}\, s + K_{df})\,\frac{z_f - z_{wf}}{b/2} \right] \tag{54}$$

---

*Refer to footnote on previous page.

There is also a roll moment attributable to horizontal motions, since the reacting forces are below the carbody C.G.. This is merely the moment arm h as shown on View A-A in Figure 6, times the force induced by the deflection of the effective lateral springs, which is the algebraic sum of the roll motion and yaw motion reflected to that position and the lateral motions of the side frames and carbody as a whole. This can be shown to be, with the addition of an effective damping term:

$$(C_{\ell f}\, s + K_{\ell f})\ h \left[y_c + x_f\, \psi_c - y_{sf} - h\, \theta_f \right] \tag{55}$$

where the term in brackets is actually the lateral deflection of the bolster, which moves laterally in conjunction with the lower end of the carbody. There is an additional moment introduced by the torsional rigidity between the two carbody sections and the difference in roll angle. With some damping assumed, this is simply

$$(C_r\, s + K_r)\ \ (\theta_a - \theta_f)$$

The roll moment due to the displacement of the C.G. of the carbody section from the longitudinal axis of the track when the carbody as a whole yaws will be neglected.

The entire roll moment equation for the fore carbody section can then be written:

$$\begin{aligned} J_f s^2\, \theta_f = {} & (C_r\, s + K_r)\ (\theta_a - \theta_f) \ + \\ & (C_{\ell f}\, s + K_{\ell f})\ h\ (y_c + x_f\, \psi_c - y_{sf} - h\, \theta_f) \ + \\ & \frac{b^2}{4} \left[(C_{sf}\, s + K_{sf})\ (\theta_F - \theta_f) + (C_{df}\, s + K_{df})\ \frac{z_f - z_{wf}}{b/2} \right] \end{aligned} \tag{56}$$

A similar equation may be developed for the aft carbody section utilizing F_{23} and F_{24} as the corresponding forces acting on the aft carbody section at the top of the vertical springs. Figure 7 illustrates these forces and other variables for this section. The sign convention for the positive direction of roll angle remains the same as for the fore section. Appropriate subscripts for certain variables, such as the side frame deflection, must be introduced to distinguish fore from aft. The equation will simply be stated without derivation when the final equations are assembled.

3.2.2 Lateral Forces Equation

Consider now the lateral forces on the carbody. The lateral force previously calculated for the roll moment equation will also act to produce lateral accelerations of the carbody. In addition the model considers the component of drawbar forces acting to produce lateral acceleration or yaw of the carbody.

From Figure 8, the lateral forces acting perpendicular to the vehicle axis are F_1 and F_2 in the direction shown. All angles will be assumed to be small and the forces of interest will be assumed to be those perpendicular to the longitudinal (x) axis.

Now

$$F_{25} = D_f \sin \phi_1 \tag{57}$$

$$F_{26} = D_a \sin \phi_2 \tag{58}$$

From the law of sines:

$$\frac{d}{\sin\psi_c} = \frac{c}{\sin\phi_2} \tag{59}$$

whence for small angles

$$\phi_2 = \frac{c}{d} \psi_c \tag{60}$$

Similarly

$$\phi_1 = \frac{x_{da} + x_{df} - c}{d} \psi_c \tag{61}$$

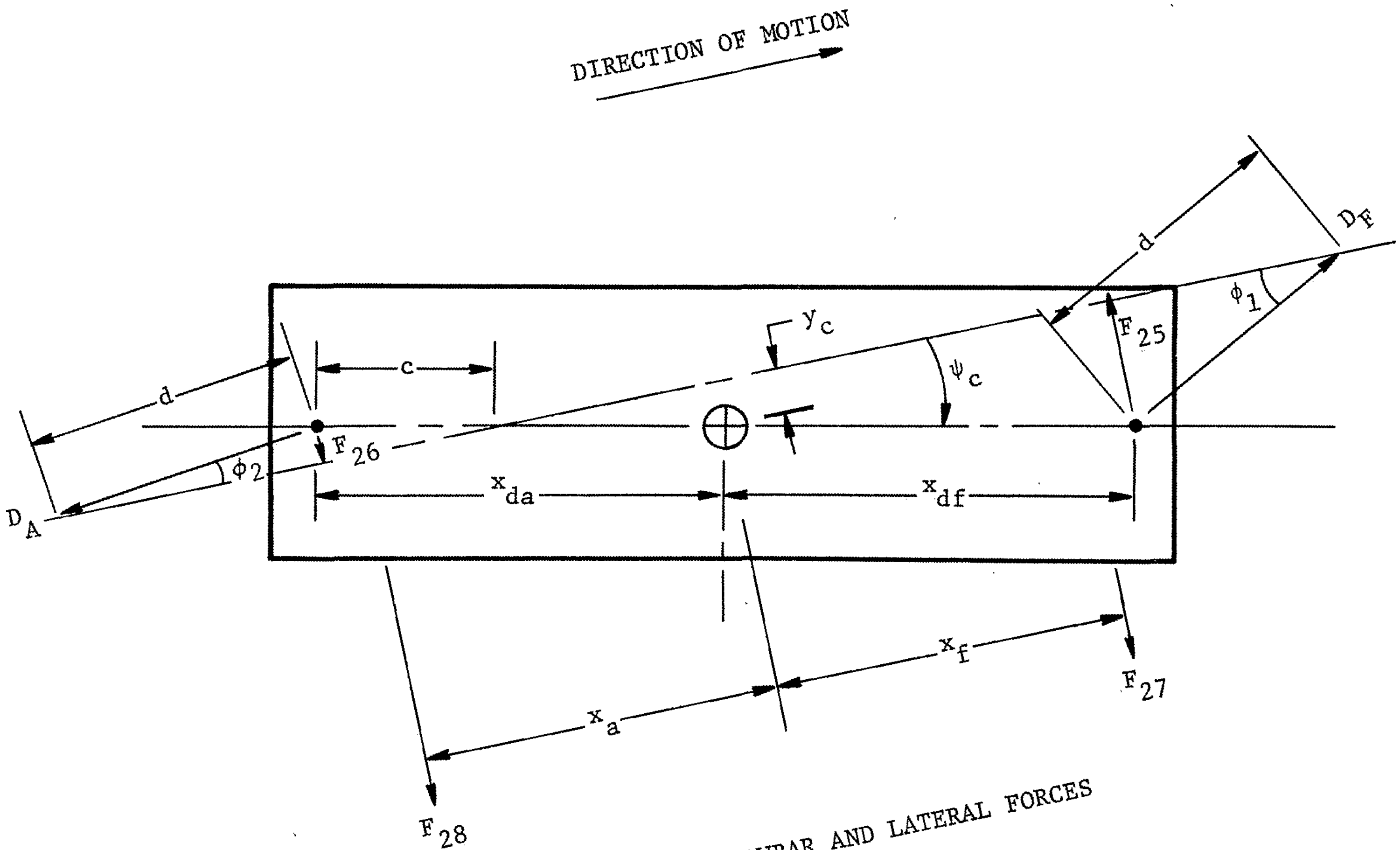


FIGURE 8. DRAWBAR AND LATERAL FORCES

Since $$\frac{y_c}{x_{da} - c} = \sin \psi_c = \psi_c \quad \text{for small angles} \tag{62}$$

by substituting into (57) and (58) we obtain

$$\phi_1 = \frac{x_{df}}{d} \psi_c + \frac{y_c}{d} \tag{63}$$

and

$$\phi_2 = \frac{x_{da}}{d} \psi_c - \frac{y_c}{d} \tag{64}$$

In turn,

$$F_{25} = \left(\frac{x_{df}}{d} \psi_c + \frac{y_c}{d} \right) D_f \tag{65}$$

$$F_{26} = \left(\frac{x_{da}}{d} \psi_c - \frac{y_c}{d} \right) D_a \tag{66}$$

By consideration of the roll moment equation and corresponding figure, we can write, by inspection:

$$F_{27} = (C_{\ell f} s + K_{\ell f}) \; (- y_c - x_f \Psi_c + h\theta_f + y_{sf}) \tag{67}$$

and

$$F_{28} = (C_{\ell a} s + K_{\ell a}) \; (- y_c + x_a \Psi_c + h\theta_a + y_{sa}) \tag{68}$$

The equation for lateral forces on the carbody is then:

$$m_c s^2 y_c = - F_{25} + F_{26} + F_{27} + F_{28} \tag{69}$$

which becomes upon substitution:

$$\begin{aligned} m_c s^2 y_c = {} & (C_{\ell f} s + K_{\ell f}) \; (- y_c - x_f \psi_c + h\theta_f + y_{sf}) + \\ & (C_{\ell a} s + K_{\ell a}) \; (- y_c + x_a \Psi_c + h\theta_a + y_{sa}) \\ & - \left(\frac{x_{df}}{d} \psi_c + \frac{y_c}{d} \right) D_f + \left(\frac{x_{da}}{d} \psi_c - \frac{y_c}{d} \right) D_a \end{aligned} \tag{70}$$

3.3.3 Yaw Moments Equation

As with the summation of lateral forces on the carbody, the summation of yaw moments may be written almost by inspection. As in lateral motion, the carbody when yawing moves as a single body.

The yaw equation in its initial form consists of terms contributed by the lateral spring forces, the forces between the bolster and the carbody contributed by the torsional restraints, and the components of drawbar forces. The initial form becomes:

$$
\begin{aligned}
J_c s^2 \psi_c = {} & F_{27} x_f - F_{28} x_a - F_{25} x_{df} - F_{26} x_{da} \\
& + (C_{cf} s + K_{cf}) (\psi_{bf} - \psi_c) \\
& + (C_{ca} s + K_{ca}) (\psi_{ba} - \psi_c)
\end{aligned}
\tag{71}
$$

where the additional subscripts "f" and "a" on the springing and damping terms and on the bolster yaw angle have been added to distinguish "fore" and "aft" terms.

Now using (65) and (66) for F_{25} and F_{26}, and (67) and (68) for F_{27} and F_{28}, with substitution the yaw equation becomes:

$$
\begin{aligned}
J_c s^2 \psi_c = {} & x_f (C_{\ell f} s + K_{\ell f}) (-y_c - x_f \psi_c + h\theta_f + y_{sf}) - \\
& x_a (C_{\ell a} s + K_{\ell a}) (-y_c + x_a \psi_c + h\theta_a + y_{sa}) \\
& + (C_{cf} s + K_{cf}) (\psi_{bf} - \psi_c) \\
& + (C_{ca} s + K_{ca}) (\psi_{ba} - \psi_c) \\
& - x_{df} \left(\frac{x_{df}}{d} \psi_c + \frac{y_c}{d} \right) D_f \\
& - x_{da} \left(\frac{x_{da}}{d} \psi_c - \frac{y_c}{d} \right) D_a
\end{aligned}
\tag{72}
$$

3.2.4 Pitch Moments Equations

The equations for the pitch moments of the two carbody sections may be simply stated by inspection of Figures 9 and 10.

Taking moments about the right end of the fore carbody and about the left end of the aft carbody, where the vertical displacements z_f and z_a are being measured, and taking consideration of inertial forces and translated inertias, results in:

$$(J_f + m_f x_{cf}^2) s^2 \alpha_f = F_s \frac{\ell}{2} - M_1 + m_f x_{cf} s^2 z_f \tag{73}$$

$$(J_a + m_a x_{ca}^2) s^2 \alpha_a = F_s \frac{\ell}{2} + M_1 - m_a x_{ca} s^2 z_a \tag{74}$$

where the shear force F_s and the bending moment M_1 are still unknowns. In addition, the relations:

$$\alpha = \alpha_f - \alpha_a \tag{75}$$

$$\alpha_f = \frac{z_f}{\ell} - \frac{z_a}{\ell} + \frac{\alpha}{2} \tag{76}$$

$$\alpha_a = \frac{z_f}{\ell} - \frac{z_a}{\ell} - \frac{\alpha}{2} \tag{77}$$

will be utilized to eliminate the intermediate variables α_f and α_a.

The bending moment must be determined as a function of α. The carbody will be modeled as a simply supported beam as shown in Figure 11 with the mass of the car lumped at the center. This is only a rough approximation as the mass is clearly distributed in an irregular fashion depending upon the carbody type and the manner in which the load is distributed. However, this approximation is regarded as no less accurate than others used in this derivation. Another approximation will be use of the slope of the beam at the ends to represent the

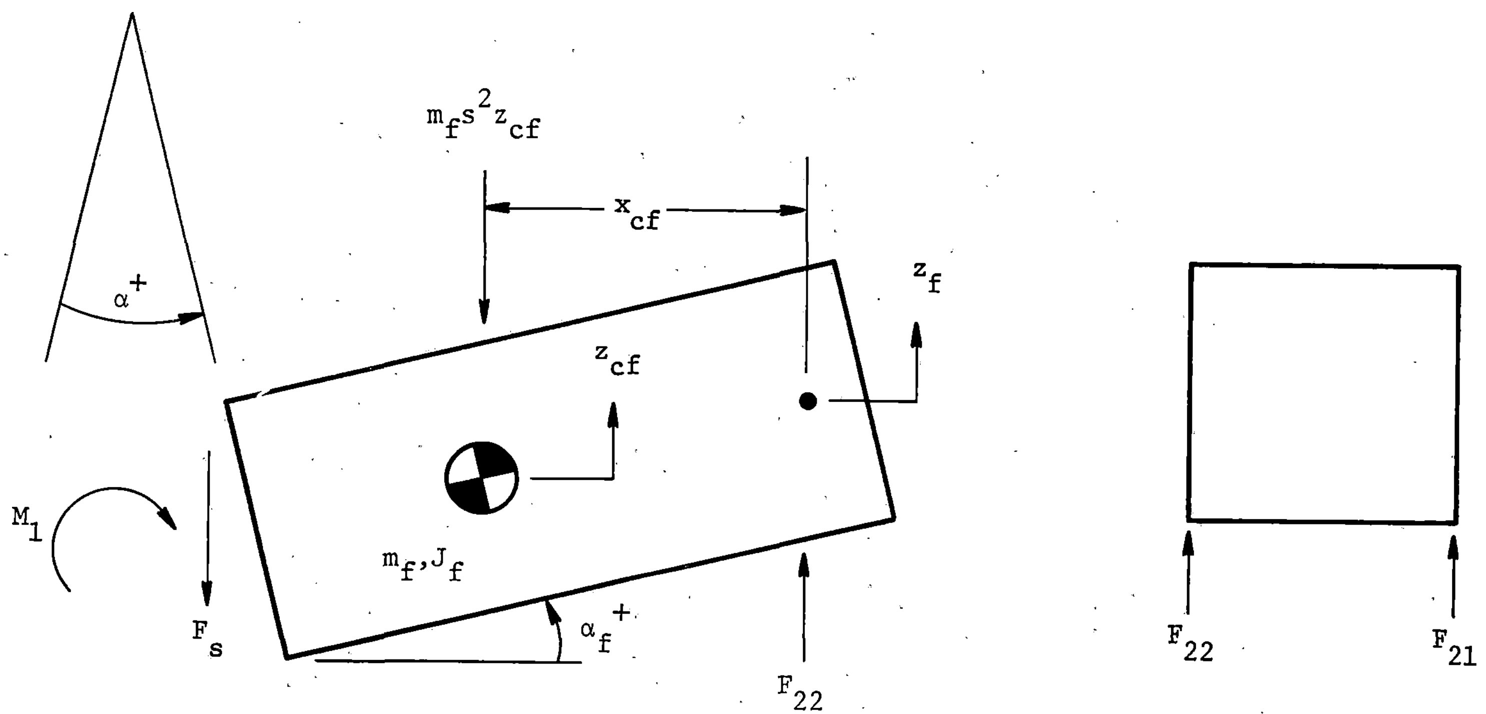


FIGURE 9. PITCH OF FORE CARBODY SECTION

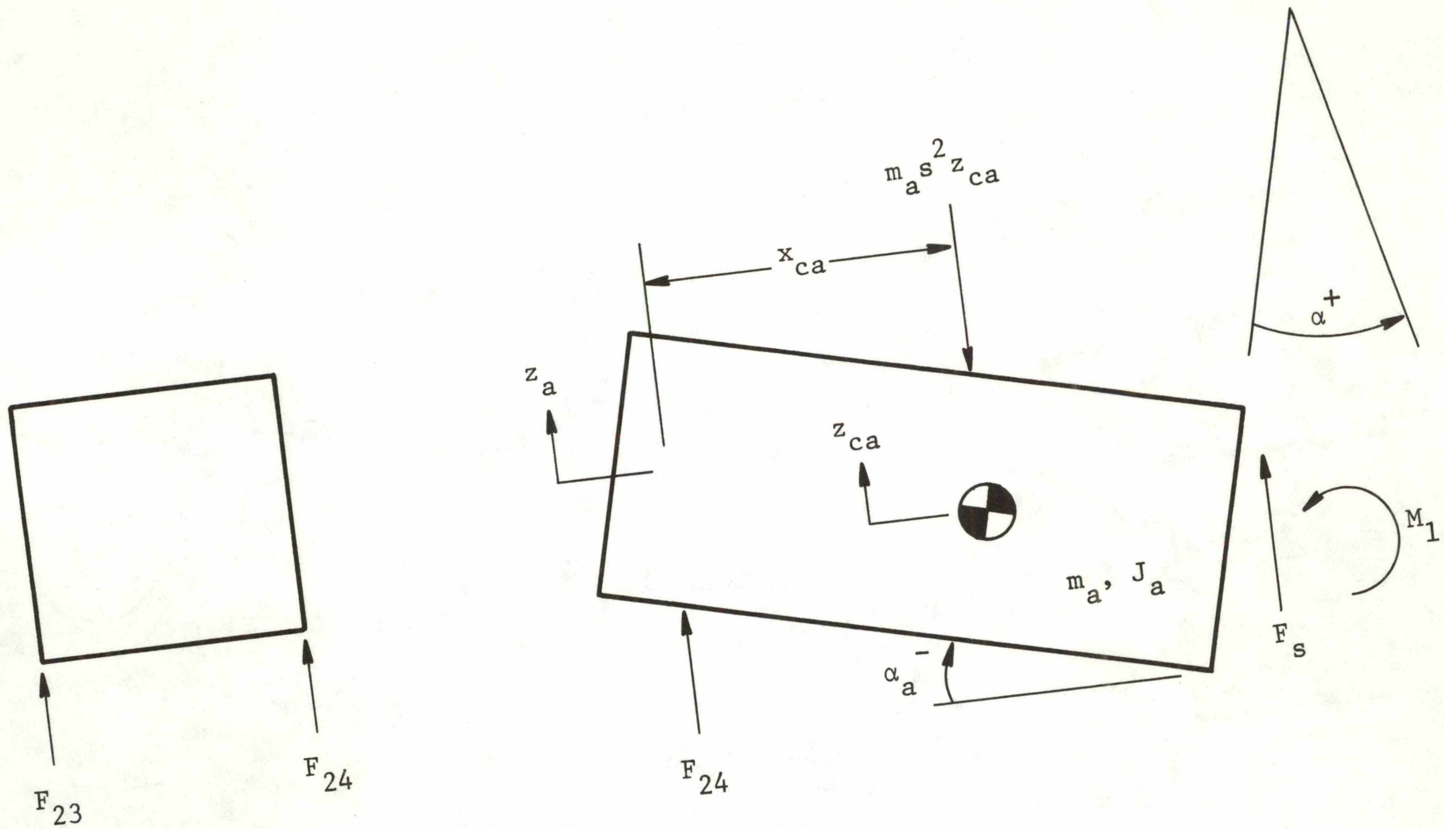


FIGURE 10. PITCH OF AFT CARBODY SECTION

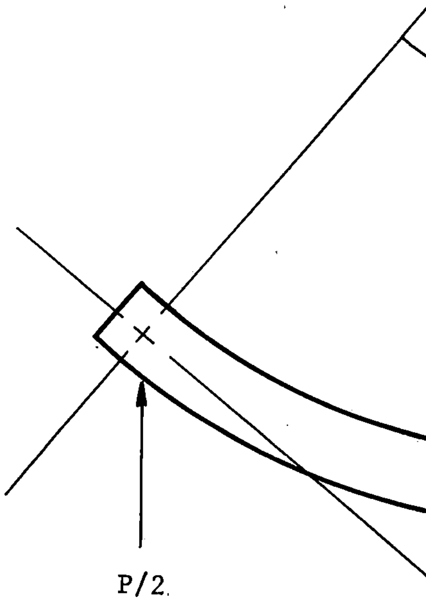
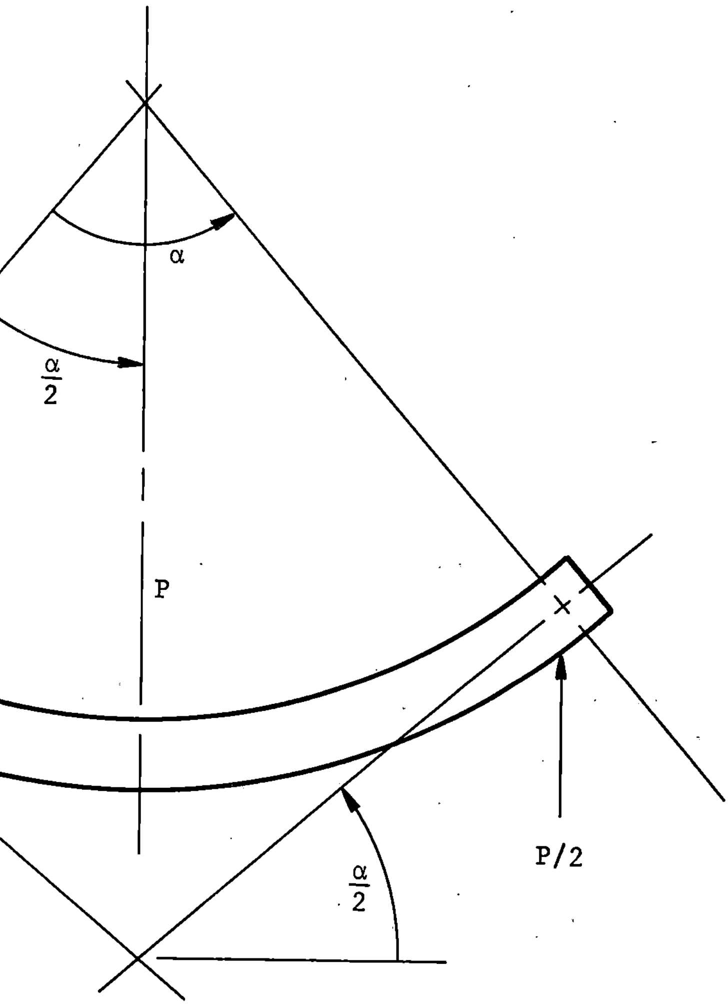


FIGURE 11.



. CARBODY BENDING DIAGRAM

angle the beam sections are bent through. With this assumption, the center angle α is clearly twice the slope of the ends, which from standard formulas is given by

$$\frac{\alpha}{2} = \frac{P\ell^2}{16EI} \tag{78}$$

The moment M_1 at the center is clearly $\frac{P}{2} \cdot \frac{\ell}{2}$. Elimination of P between these expressions permits the bending moment to be written in terms of the angle of deflection:

$$M_1 = \frac{\ell}{4} \cdot \frac{8EI\alpha}{\ell^2} = \frac{2EI\alpha}{\ell} \tag{79a}$$

Since the final equation will be written in terms of the variable α, and since we wish to assume some structural damping, however slight, define

$$K_a = \frac{2EI}{\ell} \tag{79b}$$

and C_a the corresponding damping coefficient, so that in terms of the variable of interest, M_1 becomes

$$(C_a s + K_a)\alpha \tag{79c}$$

An expression for the shear force F_s must be determined. Defining F_{21} and F_{22} as the vertical forces at the top of the bolster springs at the fore end and summing the vertical forces on the fore carbody, we obtain:

$$F_{21} + F_{22} - F_s = m_f s^2 z_{cf} \tag{80}$$

But

$$z_{cf} = z_f - x_{cf}\alpha_f \tag{81}$$

Therefore

$$F_{21} + F_{22} - F_s = m_f s^2 z_f - x_{cf} m_f s^2 \alpha_f \tag{82}$$

and $$F_s = F_{21} + F_{22} - m_f s^2 z_f + x_{cf} m_f s^2 \alpha_f \tag{83}$$

Now from previous development of the roll moment equation $F_{21} + F_{22}$ can be shown to be, in terms of some previously defined variables:

$$F_{21} + F_{22} = K_{sf} (z_{wf} - z_f) + K_{df} \frac{b}{2} (\theta_f - \theta_F) \tag{84}$$

But α_f is given by

$$\alpha_f = \frac{z_f}{\ell} - \frac{z_a}{\ell} + \frac{\alpha}{2} \tag{76}$$

and substituting in (83) yields:

$$\begin{aligned} F_s = {} & K_{sf} (z_{wf} - z_f) + K_{df} \frac{b}{2} (\theta_f - \theta_F) \\ & - \left[1 - \frac{x_{cf}}{\ell}\right] m_f s^2 z_f - \frac{x_{cf} m_f}{\ell} s^2 z_a \\ & + \frac{x_{cf} m_f}{2} s^2 \alpha \end{aligned} \tag{85}$$

Similarly, an expression for the shear force F_s may be developed from the summation of vertical forces on the aft carbody:

$$F_{23} + F_{24} + F_s = m_a s^2 z_{ca} \tag{86}$$

But

$$z_{ca} = z_a + x_{ca} \alpha_a \tag{87}$$

Therefore $$F_{23} + F_{24} + F_s = m_{ca} s^2 z_a + m_a x_{ca} s^2 \alpha_a \tag{88}$$

and $$F_s = m_a s^2 z_a + m_a x_{ca} s^2 \alpha_a - F_{23} - F_{24} \tag{89}$$

Again from the previous development of the roll moment equation, F_{23} and F_{24} can be shown, in terms of some previously defined variables, to be:

$$F_{23} + F_{24} = K_{sa} \left[z_{wa} - z_a \right] + K_{da} \frac{b}{2} \left[\theta_a - \theta_A \right] \tag{90}$$

But α_a is given by

$$\alpha_a = \frac{z_f}{\ell} - \frac{z_a}{\ell} - \frac{\alpha}{2} \tag{77}$$

and substitution into (89) yields:

$$F_s = \left[1 - \frac{x_{ca}}{\ell} \right] m_a s^2 z_a + \frac{m_a}{\ell} x_{ca} s^2 z_f - m_a x_{ca} s^2 \frac{\alpha}{2}$$

$$- K_{sa} (z_{wa} - z_a) - K_{da} \frac{b}{2} (\theta_a - \theta_A) \tag{91}$$

Substitution of the above expressions (85) and (91) for F_s, equations (76) and (77) for α_f and α_a, and the expression for M_1 into (73) and (74) will yield final equations for the pitch of each carbody section. These are, with the addition of appropriate damping terms:

For the fore carbody section,

$$\left\{ \left[\frac{J_f + m_f x_{cf}^2}{\ell} - \frac{3 x_{cf} m_f}{2} + \frac{\ell m_f}{2} \right] s^2 + \frac{\ell}{2} \left(C_{sf} s + K_{sf} \right) \right\} z_f$$

$$- \left\{ \left[\frac{J_f + m_f x_{cf}^2}{\ell} - \frac{m_f x_{cf}}{2} \right] \right\} s^2 z_a$$

$$+ \left[\frac{J_f + m_f x_{cf}^2}{2} - \frac{\ell x_{cf} m_f}{4} \right] s^2 + C_a s + K_a \Bigg\} \alpha$$

$$- \frac{\ell b}{4} K_{df} \theta_f = - \frac{\ell b}{4} K_{df} \theta_F + \frac{\ell}{2} K_{sf} z_{wf} \tag{92}$$

and for the aft carbody section,

$$\left\{ \frac{J_a + m_a x_{ca}^2}{\ell} - \frac{m_a x_{ca}}{2} \right\} s^2 z_f$$

$$- \left\{ \left[\frac{J_a + m_a x_{ca}^2}{\ell} - \frac{3m_a x_{ca}}{2} + \frac{\ell m_a}{2} \right] s^2 + \frac{\ell}{2} (C_{sa} s + K_{sa}) \right\} z_a$$

$$- \left\{ \left[\frac{J_a + m_a x_{ca}^2}{2} - \frac{\ell x_{ca} m_a}{4} \right] s^2 + C_a s + K_a \right\} \alpha$$

$$+ \frac{\ell b}{4} K_{da} \theta_a = \frac{\ell b}{4} K_{da} \theta_A - \frac{\ell}{2} K_{sa} z_{wa} \tag{93}$$

3.2.5 Vertical Forces Equation

The vertical forces on the entire carbody may be summed next. The only external forces acting on the carbody are those four shown in Figure 12. The fundamental equation becomes:

$$m_f s^2 z_{cf} + m_a s^2 z_{ca} = F_{21} + F_{22} + F_{23} + F_{24} \tag{94}$$

Recalling that

$$z_{cf} = z_f - \alpha_f x_{cf} \tag{81}$$

$$z_{ca} = z_a + \alpha_a x_{ca} \tag{87}$$

and inserting for $F_{21} + F_{22}$ and $F_{23} + F_{24}$ the values previously derived, we obtain:

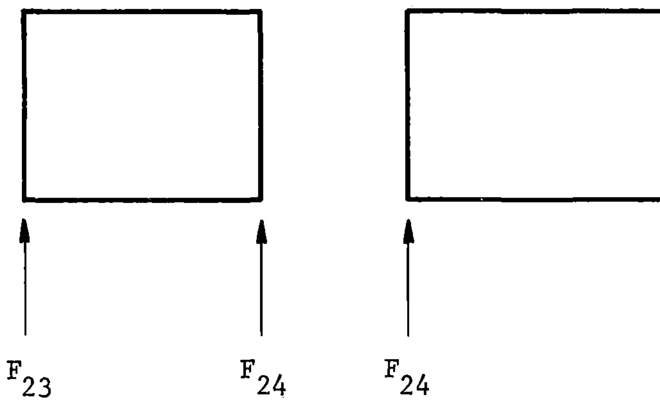
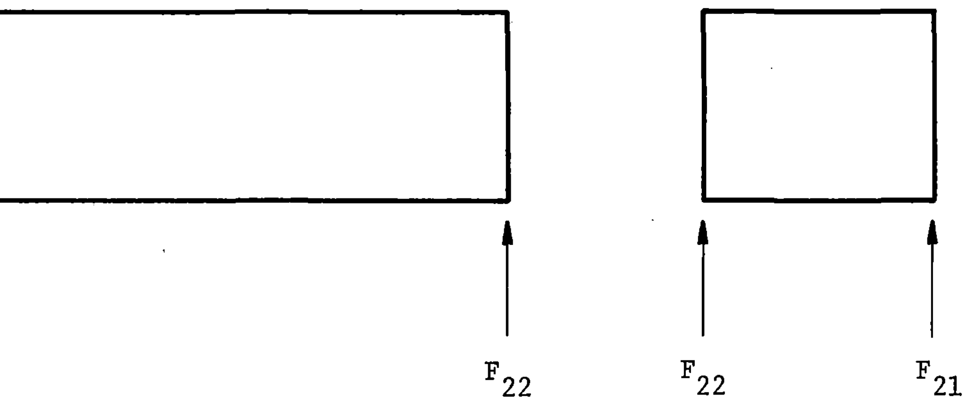


FIGURE 12.



VERTICAL FORCES DIAGRAM

$$
\begin{aligned}
&\left\{ \left[m_f \left(1 - \frac{x_{cf}}{\ell}\right) + \frac{m_a x_{ca}}{\ell} \right] s^2 + C_{sf} s + K_{sf} \right\} z_f \\
&+ \left[m_a \left(1 - \frac{x_{ca}}{\ell}\right) + \frac{m_f x_{cf}}{\ell} \right] s^2 + C_{sa} s + K_{sa} \Big\} z_a \\
&- \left[\frac{m_f x_{cf} + m_{ca} x_{ca}}{2} \right] s^2 \alpha - \left(C_{df} s + K_{df} \right) \frac{b}{2} \theta_f \\
&- \left(C_{da} s + K_{da} \right) \frac{b}{2} \theta_a = \left(C_{sf} s + K_{sf} \right) z_{wf} + \\
&\qquad \left(C_{sa} s + K_{sa} \right) z_{wa} - \\
&\qquad \left(C_{df} s + K_{df} \right) \left(\theta_A + \theta_F \right)
\end{aligned}
\tag{95}
$$

This completes the initial stages of all the equations of motion from the model.

4.0 RECAPITULATION OF EQUATIONS

The thirteen degrees of freedom require thirteen corresponding equations for complete formulation of the dynamic system. The truck equations developed in Section 2 must be rewritten for the fore and aft case. In addition, at this time, explicit recognition will be given to the equivalence of bolster lateral displacement and lateral displacement of the underside of the carbody at which point the lateral spring forces are assumed to act. This means simply that Equation 5 will be dropped in its entirety, the mass and moment of inertia of the carbody in the roll, yaw, and lateral motion equations will appropriately reflect the addition of the bolster mass, and in the truck equations the appropriate expression for displacement will replace y_5.

The corresponding thirteen output variables are:

y_{sa}, y_{sf}

ψ_{sa}, ψ_{sf}

ψ_{ba}, ψ_{bf}

θ_a, θ_f

ψ_c

y_c

z_a, z_f

α

With these considerations in mind, the thirteen equations can be simply stated, starting with the six equations from the fore and aft trucks, assuming as noted previously that the masses and moments of inertias of the truck components are identical between front and rear trucks. After some rearrangement to separate variables and place the forcing functions on the right, the equations are:

$$(m_s s^2 + C_{\ell f} s + K_{\ell f})\, y_{sf} - (C_{\ell f} s + K_{\ell f})\, y_c \tag{96}$$

$$- (C_{\ell f} s + K_{\ell f})\, x_f \psi_c + (C_{\ell f} s + K_{\ell f})\, h\theta_f = F_{9f} - F_{10f} + F_{11f} - F_{12f}$$

$$(m_s s^2 + C_{\ell a} s + K_{\ell a})\, y_{sa} - (C_{\ell a} s + K_{\ell a})\, y_c \tag{97}$$

$$+ (C_{\ell f} s + K_{\ell f})\, x_a \psi_c + (C_{\ell a} s + K_{\ell a})\, h\theta_a = F_{9a} - F_{10a} + F_{11a} - F_{12a}$$

$$(J_s s^2 + C_{bf} s + K_{bf})\, \psi_{sf} - (C_{bf} s + K_{bf})\, \psi_{bf} \tag{98}$$

$$= \frac{a}{2}\,(F_{9f} - F_{10f} - F_{11f} + F_{12f})$$

$$(J_s s^2 + C_{ba} s + K_{ba})\, \psi_{sa} - (C_{ba} s + K_{ba})\, \psi_{ba} \tag{99}$$

$$= \frac{a}{2}\,(F_{9a} - F_{10a} - F_{11a} + F_{12a})$$

$$\left[J_b s^2 + (C_{bf} + C_{cf})\, s + (K_{bf} + K_{cf})\right] \psi_{bf} - (C_{bf} s + K_{bf})\, \psi_{sf} - \tag{100}$$

$$- (C_{cf} s + K_{cf})\, \psi_c = \frac{w}{2}\,(F_{15f} - F_{16f} + F_{14f} - F_{13f})$$

$$\left[J_b s^2 + (C_{ba} + C_{ca})\, s + (K_{ba} + K_{ca})\right] \psi_{ba} - (C_{ba} s + K_{ba})\, \psi_{sa} - \tag{101}$$

$$- (C_{ca} s + K_{ca})\, \psi_c = \frac{w}{2}\,(F_{15a} - F_{16a} + F_{14a} - F_{13a})$$

$$J_f s^2 \theta_f = (C_r s + K_r)(\theta_a - \theta_f) + \qquad (56)$$

$$(C_{\ell f} s + K_{\ell f})\, h\, (y_c + x_f \psi_c - y_{sf} - h\,\theta_f) +$$

$$\frac{b^2}{4}\left[(C_{sf} s + K_{sf})(\theta_F - \theta_f) + (C_{df} s + K_{df})\left(\frac{z_f - z_{wf}}{b/2}\right)\right]$$

$$J_a s^2 \theta_a = (C_r s + K_r)(\theta_f - \theta_a) + \qquad (102)$$

$$(C_{\ell a} s + K_{\ell a})\, h\, (y_c - x_a \psi_c - y_{sa} - h\,\theta_a) +$$

$$\frac{b^2}{4}\left[(C_{sa} s + K_{sa})(\theta_A - \theta_a) + (C_{da} s + K_{da})\left(\frac{z_a - z_{wa}}{b/2}\right)\right]$$

$$m_c s^2 y_c = (C_{\ell f} s + K_{\ell f})(-y_c - x_f \psi_c + h\,\theta_f + y_{sf}) + \qquad (70)$$

$$(C_{\ell a} s + K_{\ell a})(-y_c + x_a \psi_c + h\,\theta_a + y_{sa})$$

$$-\left(\frac{x_{df}}{d}\psi_c + \frac{y_c}{d}\right) D_f + \left(\frac{x_{da}}{d}\psi_c - \frac{y_c}{d}\right) D_a$$

$$J_c s^2 \psi_c = x_f (C_{\ell f} s + K_{\ell f})(-y_c - x_f \psi_c + h\,\theta_f + y_{sf}) - \qquad (72)$$

$$x_a (C_{\ell a} s + K_{\ell a})(-y_c + x_a \psi_c + h\,\theta_a + y_{sa})$$

$$+ (C_{cf} s + K_{cf})(\psi_{bf} - \psi_c) + (C_{ca} s + K_{ca})(\psi_{ba} - \psi_c)$$

$$- x_{df}\left(\frac{x_{df}}{d}\psi_c + \frac{y_c}{d}\right) D_f - x_{da}\left(\frac{x_{da}}{d}\psi_c - \frac{y_c}{d}\right) D_a$$

The fore pitch equation is (with damping):

$$\left\{\left[\frac{J_f + m_f x_{cf}^2}{\ell} - \frac{3x_{cf} m_f}{2} + \frac{\ell m_f}{2}\right] s^2 + \frac{\ell}{2}\left(C_{sf} s + K_{sf}\right)\right\} z_f \tag{92}$$

$$-\left\{\left[\frac{J_f + m_f x_{cf}^2}{\ell} - \frac{m_f x_{cf}}{2}\right]\right\} s^2 x_a$$

$$+\left\{\left[\frac{J_f + m_f x_{cf}^2}{2} - \frac{\ell x_{cf} m_f}{4}\right] s^2 + C_a s + K_a\right\} \alpha$$

$$-\frac{\ell b}{4} K_{df}\,\theta_f = -\frac{\ell b}{4} K_{df}\,\theta_F + \frac{\ell}{2} K_{sf} z_{wf}$$

The aft pitch equation is:

$$\left\{\frac{J_a + m_a x_{ca}^2}{\ell} - \frac{m_a x_{ca}}{2}\right\} s^2 z_f \tag{93}$$

$$-\left\{\left[\frac{J_a + m_a x_{ca}^2}{\ell} - \frac{3m_a x_{ca}}{2} + \frac{\ell m_a}{2}\right] s^2 + \frac{\ell}{2}\left(C_{sa} + K_{sa}\right)\right\} z_a$$

$$-\left\{\left[\frac{J_a + m_a x_{ca}^2}{2} - \frac{\ell x_{ca} m_a}{4}\right] s^2 + C_a s + K_a\right\} \alpha$$

$$+\frac{\ell b}{4} K_{da}\,\theta_a = \frac{\ell b}{4} K_{da}\,\theta_A - \frac{\ell}{2} K_{sa}\, z_{wa}$$

The vertical force equation becomes:

$$\left\{\left[m_f\left(1-\frac{x_{cf}}{\ell}\right)+\frac{m_a x_{ca}}{\ell}\right]s^2+C_{sf}s+K_{sf}\right\}z_f \tag{95}$$

$$+\left[m_a\left(1-\frac{x_{ca}}{\ell}\right)+\frac{m_f x_{cf}}{\ell}\right]s^2+C_{sa}s+K_{sa}\Big\}\, z_a$$

$$-\left[\frac{m_f x_{cf}+m_a x_{ca}}{2}\right]s^2\alpha-\left(C_{df}s+K_{df}\right)\frac{b}{2}\,\theta_f$$

$$-\left(C_{da}s+K_{da}\right)\frac{b}{2}\,\theta_a=\left(C_{sf}s+K_{sf}\right)z_{wf}+\left(C_{sa}s+K_{sa}\right)z_{wa}-$$

$$\left(C_{df}s+K_{df}\right)\left(\theta_A+\theta_F\right)$$

It may be noted that the input and output variables have not been completely separated in these forces of the equations. This has not been done because certain terms which appear to be separate input terms implicity include the output variables. Examples of such terms are the F_9 through F_{16} lateral and longitudinal forces shown in Equations (96) through (101) which are actually complicated functions of the wheel position itself and other variables. The equations relating such input terms to the output variables are developed at some length in the contractor's document describing the model, and although these relationships are essential for a utilization of the equations developed herein, it was not the intent of this report to establish an independent mathematical model separate from the contractor's. For this reason, the reader is referred to Reference [7] for additional information regarding the input functions and their implicit relationships to some of the output variables.

5.0 SUMMARY

MITRE/METREK has taken the contractor's model of the truck and carbody and, following a development similar to the contractor's, has developed equations of motion, utilizing clearer illustrations and simpler, less computer-oriented, terminology. This report documents the effort for future reference.

The reader is referred to the contractor's documents for additional mathematical development of the forcing functions necessary to completely describe the mathematical model to be used for predicting truck behavior.

GLOSSARY OF TERMS

In cases where terms have two subscripts, the first refers to function and the second to its position, either "f" for fore carbody section, or "a" for aft carbody section. Most terms with numerical subscripts are illustrated on the various figures; many of the terms with literal subscripts are also illustrated.

C_1 = Equivalent linear lateral damping coefficient between bolster and left side frame.

C_2 = Same as C_1, except right side frame.

C_3 = Equivalent linear torsional damping coefficient between bolster and left side frame.

C_4 = Same as C_3, except right side frame.

C_a = Effective damping coefficient in carbody bending.

C_b = Equivalent linear torsional damping coefficient between bolster and side frames, also C_{bf}, C_{ba}, defined by Eq. (23).

C_c = Equivalent linear torsional damping coefficient between bolster and carbody, also C_{cf}, C_{ca}.

C_d = The difference between linear vertical damping coefficients of the left and right vertical spring groups under the bolster, also C_{da}, C_{df}, defined by Eq. (40b).

C_ℓ = Equivalent linear lateral damping coefficient between side frame and bolster (or carbody), also $C_{\ell f}$, $C_{\ell a}$, defined by Eq. (4a).

C_r = Torsional damping coefficient between fore and aft carbody sections.

C_s = The sum of the equivalent linear vertical damping coefficients of the left and right vertical spring groups under the bolster, also C_{sa}, C_{sf}, defined by Eq. (39b).

D_a = Force acting on drawbar from trailing car.

D_f = Force acting on drawbar from leading car.

E = Effective elastic modulus for entire carbody.

F_1, F_2 = Lateral forces between bolster and side frame.

F_3, F_4 = Longitudinal forces between side frames and wheelsets.

F_5, F_6 = Lateral forces between side frames and wheelsets.
F_7, F_8

F_9, F_{10} = External lateral forces acting on wheelsets.
F_{11}, F_{12}

F_{13}, F_{14} = External longitudinal forces acting on wheelsets.
F_{15}, F_{16}

F_{17}, F_{18} = Longitudinal forces between side frames and wheelsets.
F_{19}, F_{20}

F_{21}, F_{22} = Vertical forces acting on fore carbody section.

F_{23}, F_{24} = Vertical forces acting on aft carbody section.

F_{25}, F_{26} = Lateral component of drawbar forces.

F_{27}, F_{28} = Lateral forces between side frames and carbody.

F_c = Fictitious force between carbody and bolster.

I = Effective cross-sectional moment of inertia of entire carbody.

J_1 = Moment of inertia of left side frame about its C.G.

J_2 = Moment of inertia of right side frame about its C.G.

J_3 = Moment of inertia of fore wheelset about its C.G.

J_4 = Moment of inertia of aft wheelset about its C.G.

J_5 = Moment of inertia of bolster wheelset about its C.G.

J_a = Moment of inertia of aft section of carbody in pitch about C.G. of aft section, including effect of bolster mass.

J_b = Effective moment of inertia of bolster and wheelsets in yaw, about center of truck, defined by Eq. (37).

J_c = Moment of inertia of entire carbody in yaw about its C.G., not including bolster moments of inertia.

J_f = Moment of inertia of fore section of carbody in pitch about C.G. of fore section including effect of bolster mass.

J_s = Effective moment of inertia of side frames in yaw, about center of truck, defined by Eq. (22).

K_1 = Effective lateral spring constant between bolster and left side frame.

K_2 = Same as K_1, except right side frame.

K_3 = Effective torsional spring constant between bolster and left side frame.

K_4 = Same as K_3, except right side frame.

K_a = Effective carbody stiffness in bending, defined by Eq. (76b).

K_b = Effective torsional spring constant between bolster and side frames, defined by Eq. (24), also K_{bf} and K_{ba}.

K_c = Effective torsional spring constant between bolster and carbody, also K_{cf} and K_{ca}.

K_d = Difference between vertical spring constants of spring groups on each side of bolster, defined by Eq. (40a), also K_{df} and K_{da}.

K_e = Spring constant of spring group on "even" side of bolster, also K_{ef} and K_{ea}.

K_ℓ = Effective lateral spring constant between bolster (or carbody) and side frames, defined by Eq. (4b), also $K_{\ell f}$ and $K_{\ell a}$.

K_o = Same as K_e, only on "odd" side of bolster.

K_r = Effective torsional spring constant between fore and aft carbody sections.

K_s = Sum of vertical spring constants of spring groups on each side of bolster, defined by Eq. (39a), also K_{sf} and K_{sa}.

M_1 = Torsional moment between bolster and left side frame.

M_2 = Torsional moment between bolster and right side frame.

M_c = Torsional moment between bolster and right carbody as defined by Eq. (31).

a = Longitudinal distance between wheelsets in a single truck; also, a subscript meaning "aft."

b = Lateral distance between spring groups underneath bolster.

c = Distance defined on Figure 8.

d = Drawbar length.

f = Subscript meaning "fore."

ℓ = Length of entire carbody, measured between centerplates.

m_1 = Mass of left side frame.

m_2 = Mass of right side frame.

m_3 = Mass of fore wheelset.

m_4 = Mass of aft wheelset.

m_5 = Mass of bolster.

m_a = Mass of aft carbody section, including mass of aft bolster.

m_c = Mass of entire carbody, including mass of two bolsters.

m_f = Mass of fore carbody section, including mass of fore bolster.

m_s = Effective mass of side frames and wheelsets, defined by Eq. (15).

w = Lateral distance between side frames.

x_1 = Longitudinal displacement of left side frame.

x_2 = Longitudinal displacement of right side frame.

x_a = Longitudinal distance between aft centerplate and C.G. of entire carbody.

x_{ca} = Longitudinal distance between aft centerplate and C.G. of aft carbody section.

x_{cf} = Longitudinal distance between fore centerplate and C.G. of fore carbody section.

x_{da} = Longitudinal distance between aft drawbar connection and C.G. of entire carbody.

x_{df} = Longitudinal distance between fore drawbar connection and C.G. of entire carbody.

x_e = Small vertical displacement of top of spring group on "even" side of carbody, also x_{ea} and x_{ef}.

x_f = Longitudinal distance between fore centerplate and C.G. of entire carbody.

x_o = Small vertical displacement of top of spring group on "odd" side of carbody, also x_{oa} and x_{of}.

y_1 = Lateral displacement of left side frame.

y_2 = Lateral displacement of right side frame.

y_3 = Lateral displacement of fore wheelset.

y_4 = Lateral displacement of aft wheelset.

y_5 = Lateral displacement of bolster wheelset.

y_s = Lateral displacement of side frames, defined by Eq. (10), also y_{sa}, and y_{sf}.

$z_1 \ldots z_8$ = Vertical positions of wheels with corresponding subscripts.

z_a = Vertical displacement of end of aft carbody section above centerplate.

z_{ca} = Vertical displacement of C.G. of aft carbody section.

z_{cf} = Vertical displacement of C.B. of fore carbody section.

z_e = Small vertical displacement of bottom of spring group on "even" side of carbody, also z_{ea} and z_{ef}.

z_f = Vertical displacement of end of fore carbody section above centerplate, defined by Eq. (47).

z_o = Small vertical displacement of bottom of spring group on "odd" side of carbody, also z_{of} and z_{oa}.

z_w = Average vertical input, also z_{wa}, z_{wf} defined by Eq. (49).

α = Angle of bend between carbody sections, defined by Fig. 3 and Eq. (75).

α_a = Pitch angle of aft carbody section.

α_f = Pitch angle of fore carbody section.

θ_A = Roll angle of hypothetical line joining bottom of vertical spring groups on aft truck.

θ_a = Roll angle of aft carbody section.

θ_F = Same as θ_A, only for fore truck, defined by Eq. (48).

θ_f = Roll angle of fore carbody section, defined by Eq. (46).

ϕ_1 = Angle between fore drawbar and x axis.

ϕ_2 = Angle between aft drawbar and x axis.

ψ_b = Absolute yaw angle of bolster, also ψ_{ba}, ψ_{bf}.

ψ_c = Absolute yaw angle of entire carbody.

ψ_s = Absolute yaw angle of side frames, also ψ_{sa}, ψ_{sf}.

$\triangleq$ = Is equal by definition to; or, is defined as.

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